# RESTRICTION OF TORAL EIGENFUNCTIONS TO HYPERSURFACES AND NODAL SETS

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**Abstract.** We give uniform upper and lower bounds for the $L^2$ norm of the restriction of eigenfunctions of the Laplacian on the three-dimensional standard flat torus to surfaces with non-vanishing curvature. We also present several related results concerning the nodal sets of eigenfunctions.

## 1 Introduction

Let $M$ be a smooth Riemannian surface without boundary, $\Delta$ the corresponding Laplace–Beltrami operator and $\Sigma$ a smooth curve in $M$. Burq, Gérard and Tzvetkov [BGT07] established bounds for the $L^2$-norm of the restriction of eigenfunctions of $\Delta$ to the curve $\Sigma$, showing that if $-\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$, $\lambda > 0$, then

$$||\varphi_\lambda||_{L^2(\Sigma)} \ll \lambda^{1/4}||\varphi_\lambda||_{L^2(M)} \tag{1.1}$$

and if $\Sigma$ has non-vanishing geodesic curvature then (1.1) may be improved to

$$||\varphi_\lambda||_{L^2(\Sigma)} \ll \lambda^{1/6}||\varphi_\lambda||_{L^2(M)}. \tag{1.2}$$

Both (1.1), (1.2) are saturated for the sphere $S^2$.

In [BGT07] it is observed that for the flat torus $M = \mathbb{T}^2$, (1.1) can be improved to

$$||\varphi_\lambda||_{L^2(\Sigma)} \ll \lambda^\epsilon||\varphi_\lambda||_{L^2(M)}, \quad \forall\epsilon > 0 \tag{1.3}$$

due to the fact that there is a corresponding bound on the supremum of the eigenfunctions. They raise the question whether in (1.3) the factor $\lambda^\epsilon$ can be replaced by a constant, that is whether there is a uniform $L^2$ restriction bound. As pointed out by Sarnak [Sar], if we take $\Sigma$ to be a geodesic segment on the torus, this particular problem is essentially equivalent to the currently open question of whether on the circle $|x| = \lambda$, the number of lattice points on an arc of size $\lambda^{1/2}$ admits a uniform bound.

In [BGT07] results similar to (1.1) are also established in the higher dimensional case for restrictions of eigenfunctions to smooth submanifolds, in particular (1.1)

holds for codimension-one submanifolds (hypersurfaces) and is sharp for the sphere $S^{d-1}$. Moreover (1.2) remains valid for hypersurfaces with positive curvature [Hu09].

In this paper we pursue the improvements of (1.2) for the standard flat $d$-dimensional tori $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, considering the restriction to (codimension-one) hypersurfaces $\Sigma$ with non-vanishing curvature.

**Main Theorem** Let $d = 2, 3$ and let $\Sigma \subset \mathbb{T}^d$ be a real analytic hypersurface with non-zero curvature. There are constants $0 < c < C < \infty$ and $\Lambda > 0$, all depending on $\Sigma$, so that all eigenfunctions $\varphi_\lambda$ of the Laplacian on $\mathbb{T}^d$ with $\lambda > \Lambda$ satisfy

$$c||\varphi_\lambda||_2 \leq ||\varphi_\lambda||_{L^2(\Sigma)} \leq C||\varphi_\lambda||_2. \tag{1.4}$$

Observe that for the lower bound, the curvature assumption is necessary, since the eigenfunctions $\varphi(x) = \sin(2\pi n_1 x_1)$ all vanish on the hypersurface $x_1 = 0$. In fact this lower bound implies that a curved hypersurface cannot be contained in the nodal set of eigenfunctions with arbitrarily large eigenvalues.

It was shown in [BR11a] that this last property of the nodal sets of toral eigenfunctions hold in arbitrary dimension $d$. As we point out in Sect. 10, the argument from [BR11a] implies in fact a bound for the $d - 2$ dimensional Hausdorff measure of the intersection of nodal sets with a fixed hypersurface $\Sigma$:

**Theorem 1.1.** *Let $\Sigma \subset \mathbb{T}^d$ be a real analytic hypersurface with nowhere vanishing curvature. Then for $\lambda > \lambda_\Sigma$, the nodal set $N$ of any eigenfunction $\varphi_\lambda$ satisfies*

$$h_{d-2}(N \cap \Sigma) < c_\Sigma \lambda. \tag{1.5}$$

For dimension $d = 2$, this means an upper bound for the number of intersection points of a fixed curve with the nodal lines. Interestingly, using the Main Theorem, one can show that conversely:

**Theorem 1.2.** *Let $\Sigma \subset \mathbb{T}^2$ be a real analytic non-geodesic curve. There is $\lambda_\Sigma$ such that for $\lambda > \lambda_\Sigma$, the nodal set $N$ of any eigenfunction $\varphi_\lambda$ satisfies*

$$\#(N \cap \Sigma) \gg \lambda^{1-\varepsilon} \quad \text{for all } \varepsilon > 0 \tag{1.6}$$

and for $d = 3$, the following property

**Theorem 1.3.** *Let $\Sigma \subset \mathbb{T}^3$ be as in the Main Theorem. There is $\lambda_\Sigma$ such that for $\lambda > \lambda_\Sigma$, the nodal set $N$ of any eigenfunction $\varphi_\lambda$ intersects $\Sigma$.*

Returning to the results of [BGT07] for smooth Riemannian surfaces, let us point out that there is a close connection between estimates on $||\varphi_\lambda||_{L^2(\Sigma)}$ with $\Sigma$ a geodesic segment and bounds on the $L^4$-norm $||\varphi_\lambda||_{L^4(M)}$. Recall Sogge's general estimate for the $L^p$-norm [Sog88]

$$||\varphi_\lambda||_{L^p(M)} \leq C\lambda^{\delta(p)}||\varphi_\lambda||_{L^2(M)} \tag{1.7}$$

where

$$\delta(p) = \begin{cases} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & \text{if } 2 \le p \le 6 \\ \frac{1}{2} - \frac{2}{p} & \text{if } 6 \le p \le \infty. \end{cases} \tag{1.8}$$

The following inequalities were established in [Bou09]

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \le C\lambda^{\frac{1}{2p}} \|\varphi_\lambda\|_{L^p(M)} \tag{1.9}$$

if $\Sigma \subset M$ is a geodesic segment and $p \ge 2$, and conversely

$$\|\varphi_\lambda\|_{L^4(M)} \ll \lambda^{\frac{1}{16}+\varepsilon} \max_\Sigma \|\varphi_\lambda\|_{L^2(\Sigma)}^{\frac{1}{4}}. \tag{1.10}$$

where the maximum is over all geodesic segments $\Sigma \subset M$ of unit length. Hence (1.9), (1.10) imply that improving upon the restriction bound (1.1) is essentially equivalent with convexity breaking for the $L^4$-norm (see also [Sog11]). Of course for $M = \mathbb{T}^2$, $\|\varphi_\lambda\|_\infty \ll \lambda^\varepsilon$ and previous considerations are of no interest. However, the example of an integrable torus $M$ constructed in [Bou93] provides a sequence of eigenfunctions $\varphi_\lambda$ and a geodesic segment $\Sigma \subset M$ such that

$$\|\varphi_\lambda\|_{L^6(M)} \sim \lambda^{\frac{1}{6}} \quad \text{and} \quad \|\varphi_\lambda\|_{L^2(\Sigma)} \sim \lambda^{\frac{1}{4}}. \tag{1.11}$$

Thus this example saturates the inequality (1.9) for $p = 6$ and also the [BGT07] bound (1.1) (providing a surface quite different from the sphere).

The proof of the Main Theorem for $d = 2$ is rather simple (compared with $d = 3$) and we describe it next, as an illustration of the method and some of the arithmetic ingredients used, see [BR09].

Denote by $\sigma$ the normalized arc-length measure on the curve $\Sigma$. Using the method of stationary phase, one sees that if $\Sigma$ has non-vanishing curvature then the Fourier transform $\widehat{\sigma}$ decays as

$$|\widehat{\sigma}(\xi)| \ll |\xi|^{-1/2}, \quad \xi \ne 0. \tag{1.12}$$

Moreover $|\widehat{\sigma}(\xi)| \le \widehat{\sigma}(0) = 1$ with equality only for $\xi = 0$, hence

$$\sup_{0 \ne \xi \in \mathbb{Z}^2} |\widehat{\sigma}(\xi)| \le 1 - \delta, \tag{1.13}$$

for some $\delta = \delta_\Sigma > 0$.

An eigenfunction of the Laplacian on $\mathbb{T}^2$ is a trigonometric polynomial of the form:

$$\varphi(x) = \sum_{|n|=\lambda} \widehat{\varphi}(n)e(n \cdot x)$$

(where $e(z) := e^{2\pi i z}$), all of whose frequencies lie in the set $\mathcal{E} := \mathbb{Z}^2 \cap \lambda S^1$. As is well known, in dimension $d = 2$, $\#\mathcal{E} \ll \lambda^\epsilon$ for all $\epsilon > 0$. Moreover, by a result of Jarnik [Jar26], any arc on $\lambda S^1$ of length at most $c\lambda^{1/3}$ contains at most two lattice points

(Cilleruelo and Cordoba [CC92] showed that for any $\delta < \frac{1}{2}$, arcs of length $\lambda^\delta$ contain at most $M(\delta)$ lattice points and in [CG07] it is conjectured that this remains true for any $\delta < 1$). Hence we may partition,

$$\mathcal{E} = \bigcup_\alpha \mathcal{E}_\alpha \tag{1.14}$$

where $\#\mathcal{E}_\alpha \leq 2$ and $\text{dist}(\mathcal{E}_\alpha, \mathcal{E}_\beta) > c\lambda^{1/3}$ for $\alpha \neq \beta$. Correspondingly we may write,

$$\varphi = \sum_\alpha \varphi^\alpha, \quad \varphi^\alpha(x) = \sum_{n \in \mathcal{E}_\alpha} \widehat{\varphi}(n)e(nx), \tag{1.15}$$

so that $||\varphi||_2^2 = \sum_\alpha ||\varphi^\alpha||_2^2$ and

$$\int_\Sigma |\varphi|^2 d\sigma = \sum_\alpha \sum_\beta \int_\Sigma \varphi^\alpha \overline{\varphi^\beta} d\sigma. \tag{1.16}$$

Applying (1.12) we see that $\int_\Sigma \varphi^\alpha \overline{\varphi^\beta} d\sigma \ll \lambda^{-1/6}$ if $\alpha \neq \beta$ and because $\#\mathcal{E} \ll \lambda^\epsilon$ the total sum of these nondiagonal terms is bounded by $\lambda^{-1/6+\epsilon}||\varphi||_2^2$. It suffices then to show that the diagonal terms satisfy

$$\delta||\phi^\alpha||_2^2 \leq \int_\Sigma |\phi^\alpha|^2 d\sigma \leq 2||\phi^\alpha||_2^2. \tag{1.17}$$

This is clear if $\mathcal{E}_\alpha = \{n\}$ while if $\mathcal{E}_\alpha = \{m, n\}$ then

$$\int_\Sigma |\phi^\alpha|^2 d\sigma = |\widehat{\varphi}(m)|^2 + |\widehat{\varphi}(n)|^2 + 2\text{Re } \widehat{\varphi}(m)\overline{\widehat{\varphi}(n)}\widehat{\sigma}(m - n), \tag{1.18}$$

and then (1.17) follows from (1.13) This proves the Theorem for $d = 2$.

The proof of the Main Theorem for dimension $d = 3$ is considerably more involved and occupies Sects. 2–9 of the paper. Arguing along the lines of the two-dimensional case gives an upper bound of $\lambda^\epsilon$. To get the uniform bound for $d = 3$ we need to replace the upper bound (1.12) for the Fourier transform of the hypersurface measure by an asymptotic expansion, and then exploit cancellation in the resulting exponential sums over the sphere. A key ingredient there is controlling the number of lattice points in spherical caps.

To state some relevant results, denote as before by $\mathcal{E} = \mathbb{Z}^d \cap \lambda S^{d-1}$ the set of lattice points on the sphere of radius $\lambda$. We have $\#\mathcal{E} \ll \lambda^{d-2+\epsilon}$. Let $F_d(\lambda, r)$ be the maximal number of lattice points in the intersection of $\mathcal{E}$ with a spherical cap of size $r > 1$. A higher-dimensional analogue of Jarnik's theorem implies that if $r \ll \lambda^{1/(d+1)}$ then all lattice points in such a cap are co-planar, hence $F_d(r, \lambda) \ll r^{d-3+\epsilon}$ in that case, for any $\epsilon > 0$. For larger caps, we show:

PROPOSITION 1.4.  (i) Let $d = 3$. Then for any $\eta < \frac{1}{15}$,

$$F_3(\lambda, r) \ll \lambda^\epsilon \left( r \left( \frac{r}{\lambda} \right)^\eta + 1 \right). \tag{1.19}$$

(ii) Let $d = 4$. Then

$$F_4(\lambda, r) \ll \lambda^\epsilon \left( \frac{r^3}{\lambda} + r^{3/2} \right). \tag{1.20}$$

(iii) For $d \geq 5$ we have

$$F_d(\lambda, r) \ll \lambda^\epsilon \left( \frac{r^{d-1}}{\lambda} + r^{d-3} \right) \tag{1.21}$$

(the factor $\lambda^\epsilon$ is redundant for large $d$).

The term $r^{d-1}/\lambda$ concerns the equidistribution of $\mathcal{E}$, while the term $r^{d-3}$ measures deviations related to accumulation in lower dimensional strata.

Only (1.19) ($d = 3$) is relevant for our purpose (Lemma 6.8 in the paper, proved in Section 9) and (1.20), (1.21) for $d \geq 4$ (proven in Appendix A.) were included to provide a more complete picture. We point out that the argument used to obtain (1.19) is based on certain diophantine considerations and dimension reduction, hence differs considerably from the proof of (1.20), (1.21) using standard Hardy–Littlewood circle method and Kloosterman's refinement for $d = 4$.

The second result expresses a mean-equidistribution property of $\mathcal{E}$. Partition the sphere $\lambda S^2$ into sets $C_\alpha$ of size $\lambda^{1/2}$, for instance by intersecting with cubes of that size. Since $\#\mathcal{E} \ll \lambda^{1+\epsilon}$, one may expect that $\#C_\alpha \cap \mathcal{E} \ll \lambda^\epsilon$. We show (in joint work with Sarnak [BRS]) that as a consequence of "Linnik's basic Lemma", this holds in the mean square:

LEMMA 1.5.

$$\sum_\alpha [\#(\mathcal{E} \cap C_\alpha)]^2 \ll \lambda^{1+\epsilon}, \quad \forall \epsilon > 0. \tag{1.22}$$

Finally, considering very large caps $r > \lambda^{1-\delta}$, there is an estimate

LEMMA 1.6.

$$\#(\mathcal{E} \cap C_r) \ll \left( \frac{r}{\lambda} \right)^2 \lambda^{1+\varepsilon} \quad \text{for } r > \lambda^{1-\delta_0} \tag{1.23}$$

($\delta_0 > 0$ some absolute constant).

which is a consequence of Linnik's equidistribution property (see Sect. 2.1). While we make essential use of Lemma 1.5 in our analysis, Lemma 1.6 will not be needed, strictly speaking.

Let $1 < r < \lambda$ and let $C$, $C'$ be spherical $r$-caps on $\lambda S^2$ of mutual distance at least $10r$. Following the argument for $d = 2$, we need to bound exponential sums of the form

$$\sum_{n \in C} \sum_{n' \in C'} \widehat{\varphi}(n) \overline{\widehat{\varphi}(n')} e(\psi(n - n')), \qquad ||\varphi||_2 = 1 \tag{1.24}$$

where $\psi$ is the support function of the hyper-surface $\Sigma$, which appears in the asymptotic expansion of the Fourier transform of the surface measure on $\Sigma$, see Sect. 3. For instance, in the case that $\Sigma = \{|x| = 1\}$ is the unit sphere then $h(\xi) = |\xi|$.

For $r < \lambda^{1-\epsilon}$ we simply estimate (1.24) by $F_3(\lambda, r)$ (see (1.19)). When $\lambda^{1-\epsilon} < r < \lambda$ this bound does not suffice and we need to exploit cancellation in the sum (1.24).

LEMMA 1.7. *There is $\delta > 0$ so that (1.24) admits a bound of $\lambda^{1-\delta}$ for $\lambda \gg 1$.*

This statement depends essentially on the equidistribution of $\mathcal{E}$ in caps of size $\sqrt{\lambda}$, as expressed in Lemma 1.5.

Using Taylor expansions of the function $\psi(x - y)$ with $x, y$ restricted to $S^2$ and suitable coordinate restrictions, Lemma 1.7 is eventually reduced to the following one-dimensional exponential sum estimate (proven in Sect. 6):

LEMMA 1.8. *Let $\beta \gg 1$ and $X, Y \subset [0, 1]$ arbitrary discrete sets such that $|x - x'|, |y - y'| > \beta^{-1/2}$ for $x \neq x' \in X$ and $y \neq y' \in Y$. Then*

$$\left| \sum_{x \in X} \sum_{y \in Y} e(\beta x y + \beta^{1/3} x^2 y^2) \right| \ll \beta^{1-\kappa} \tag{1.25}$$

*for some $\kappa > 0$.*

Extending the Main Theorem to arbitrary dimension $d$ remains unsettled at this point. We make the following

CONJECTURE 1.9. *Let $d \geq 2$ be arbitrary and $\Sigma \subset \mathbb{T}^d$ a real analytic hypersurface. Then, for some constant $C_\Sigma$, all eigenfunctions $\varphi_\lambda$ of $\mathbb{T}^d$ satisfy*

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \leq C_\Sigma \|\varphi_\lambda\|_2. \tag{1.26}$$

*If moreover $\Sigma$ has nowhere vanishing curvature and $\lambda > \lambda_\Sigma$, for some $c_\Sigma > 0$, also*

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \geq c_\Sigma \|\varphi_\lambda\|_2. \tag{1.27}$$

It should be pointed out that in our proof of the Main Theorem for $d = 2, 3$, only distributional properties of $\mathcal{E} = \mathbb{Z}^d \cap [|x| = \lambda]$ were exploited, but not the fact that $\mathcal{E}$ actually consists of lattice points. In Sect. 11, we give an example, for $d \geq 8$, of sets $\mathcal{S}_\lambda \subset \lambda S^{d-1}$ satisfying the 'ideal' distributional property

$$|x - y| \gtrsim \lambda^{\frac{1}{d-1}} \quad \text{for } x \neq y \text{ in } \mathcal{S}_\lambda \tag{1.28}$$

and such that the Fourier restriction operator

$$L^2(S^{d-1}, d\sigma) \longrightarrow \ell^2(\mathcal{S}_\lambda) : \mu \mapsto \hat{\mu}|_{\mathcal{S}_\lambda} \tag{1.29}$$

has unbounded norm for $\lambda \to \infty$. This illustrates the difficulty for carrying out our analysis in larger dimension.

As said earlier, even for $d = 2$ and $\Sigma$ a straight line segment in $\mathbb{T}^2$, (1.26) remains open and is roughly equivalent with the arithmetic statement that the number of lattice points on an arc of size $\sqrt{\lambda}$ on the circle $|x| = \lambda$ is bounded by an absolute constant. An easy argument in [BR11b] shows that this last property is true for most $E = \lambda^2$ and in fact the elements of $\{|x|^2 = E\}$ are at least $\gg \lambda^{1-\varepsilon}$ separated, for all $\varepsilon > 0$. In Sect. 12, we establish the following

**Theorem 1.10.** *Let $\Sigma \subset \mathbb{T}^2$ be a smooth curve. Then for almost all $E = \lambda^2$, there is a uniform restriction bound*

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \leq C \|\varphi_\lambda\|_2. \tag{1.30}$$

In Sect. 13 we obtain an analogue for $\mathbb{T}^d$, $d \geq 3$ of a theorem of Nazarov and Sodin [NS09] on the number of nodal domains.

**Theorem 1.11.** *Let $d \geq 3$ and $E = \lambda^2$ be sufficiently large. Then for a 'typical' element $\varphi_\lambda$ of the eigenfunction space $-\Delta\varphi = E\varphi$, the nodal set $N$ has $\sim \lambda^d$ components.*

Recalling Courant's nodal domain theorem, the interest of Theorem 1.11 is the lower bound on the number of nodal domains.

Almost all the subsequent analysis in the paper relates to $d = 3$ and $\mathbb{T}^3$-eigenfunctions. Let us stress again that the arithmetic structure of the frequencies of the trigonometric polynomials involved is essential here.

## 2    Lattice Points in Spherical Caps

**2.1    Lattice points on spheres.**    We recall what is known concerning the total number $\rho_d(R^2)$ of lattice points on the sphere of radius $R$. Throughout we assume, as we may, that $n := R^2$ is an integer. We have a general upper bound

$$\rho_d(R^2) \ll R^{d-2+\epsilon}, \quad \forall \epsilon > 0 \tag{2.1}$$

and in dimension $d \geq 5$ we in fact have both a lower and upper bound of this strength:

$$\rho_d(R^2) \approx R^{d-2}, \quad d \geq 5. \tag{2.2}$$

In smaller dimensions both the lower and upper bound (2.1) need not hold. For instance if $n = 2^k$ is a power of 2 then $\rho_4(R^2) = 24$ is bounded. The situation in dimension $d = 3$ is particularly delicate. It is known that $\rho_3(n) > 0$ if and

only if $n := R^2 \neq 4^k(8m - 1)$. There are *primitive* lattice points on the sphere of radius $R = \sqrt{n}$ (that is $x = (x_1, x_2, x_3)$ with $\gcd(x_1, x_2, x_3) = 1$) if an only if $n \neq 0, 4, 7 \bmod 8$. Concerning the number $\rho_3(R^2)$ of lattice points, the upper bound (2.1) is still valid, and if there are primitive lattice points then there is a lower bound of $\rho_3(R^2) \gg R^{1-o(1)}$ but there are arbitrarily large $R$'s so that

$$\rho_3(R^2) \gg R \log \log R. \tag{2.3}$$

A fundamental result conjectured by Linnik (and proved by him assuming the Generalized Riemann Hypothesis), that for $n \neq 0, 4, 7 \bmod 8$, the projections of these lattice points to the unit sphere become uniformly distributed on the unit sphere as $n \to \infty$. This was proved unconditionally by Duke [Duk88, DS90] and Golubeva and Fomenko [GF90], following a breakthrough by Iwaniec [Iwa87].

**2.2   Lattice points in spherical caps: Statement of results.**      Let $\vec{\zeta} \in S^{d-1}$ be a unit vector, $R \gg 1$, and $r = o(R)$. Consider the spherical cap $C = C(R\vec{\zeta}, r)$ which is the intersection of the sphere $|\vec{x}| = R$ with the ball of radius $\approx r$ around $R\vec{\zeta}$. Set

$$F_d(R, r) = \max_{\vec{\zeta} \in S^{d-1}} \#\mathbb{Z}^d \cap C(R\vec{\zeta}, r)$$

which is the maximal number of lattice points in a spherical cap of size $r$ on the sphere $|\vec{x}| = R$. We want to give an upper bound for $F_d(R, r)$ in the case of dimension $d = 3$. The results which will be proven in this section are as follows:

(i) For all $\epsilon > 0$,

$$F_3(R, r) \ll R^\epsilon \Big(1 + \frac{r^2}{R^{1/2}}\Big). \tag{2.4}$$

This is an immediate consequence of a Jarnik-type result on non-coplanar lattice points in small caps. It is only useful for small caps, when $r \ll R^{1/2}$.

For larger caps we shall show the following bound:

(ii) For any $\eta < \frac{1}{15}$,

$$F_3(R, r) \ll R^\epsilon \Big(1 + r\Big(\frac{r}{R}\Big)^\eta\Big). \tag{2.5}$$

It is natural to conjecture that $F_3(R, r) \ll R^\epsilon\big(1 + \frac{r^2}{R}\big)$ for $r < R^{1-\delta}$.

**2.3   Intersections with hyperplanes.**      Let $\kappa_d(R)$ be the maximal number of lattice points in the intersection of the sphere $|\vec{x}| = R$ in $\mathbb{R}^d$ and a hyperplane.

For dimension $d = 2$,

$$\kappa_2(R) \leq 2$$

while in dimension $d = 3$ we have

$$\kappa_3(R) \ll R^\epsilon, \quad \forall \epsilon > 0. \tag{2.6}$$

### 2.4 Small caps.

LEMMA 2.1. *For a spherical cap $C$ of size $r$ on the sphere of radius $R$ in $\mathbb{R}^3$ the number of lattice points in $C$ is at most*

$$\#C \cap \mathbb{Z}^3 \ll R^\epsilon \Big(1 + \frac{r^2}{R^{\frac{1}{2}}}\Big). \tag{2.7}$$

*Proof.* Firstly, we note that if the cap has radius $r \ll R^{1/4}$ then it contains only a $O(R^\epsilon)$ lattice points. This can be deduced from Jarnik's method [Jar26] and also from a general result of Andrews [And63] that if $C$ is any convex body in $\mathbb{R}^d$ with volume $V$ then the number of lattice points on its boundary which are not coplanar is $\ll V^{\frac{d-1}{d+1}}$. In our case of a cap in dimension 3, the base of the cap has area $\approx r^2$ and if $\theta$ is the opening of the cap, so that $r \approx R\theta$, then the height of the cap is about $R - R\cos\theta \approx R\theta^2 \approx r^2/R$, hence the volume of the cap is $V \approx r^4/R$. Thus if $r < R^{1/4}$ then any such cap will contain at most (say) 100 non-coplanar lattice points. Any lattice points in the cap will lie on one of the plane sections of the cap through any three of the 100 non-coplanar lattice points. Each such plane section will contain at most $R^\epsilon$ lattice points (uniformly as a function of the plane) and hence the cap will contain at most $O(R^\epsilon)$ lattice points.

Now, for a cap $C$ of radius $r \geqq R^{1/4}$, divide it into caps of radius $R^{1/4}$; the number of such caps will be $\approx \operatorname{area}(C)/(R^{1/4})^2 \approx r^2/R^{1/2}$, and hence the total number of lattice points in $C$ is at most $R^\epsilon(1 + r^2/R^{1/2})$. □

### 2.5 A linear and sub-linear bound.

We now turn to larger caps. Here is a simple bound via slicing, using the fact that we can control the number of lattice points in the intersection of a sphere and a hyperplane parallel to one of the coordinate hyperplanes:

LEMMA 2.2. *In dimension $d \geq 2$,*

$$F_d(R, r) \ll (1 + r)\kappa_d(R). \tag{2.8}$$

*Proof.* A ball of radius $r$ is contained in a vertical slab of the form $A < x_d < A + 2r$ and hence all integer points in the intersection of the sphere $|\vec{x}| = R$ and the ball $|\vec{x} - \vec{x}_0| < r$ lie in the union of the planes $x_d = k$, $A \leq k \leq A + 2r$ with $k$ integer. The intersection of each plane and the sphere $|\vec{x}| = R$ has at most $\kappa_d(R)$ lattice points, and therefore the total number of lattice points is at most $(1 + r)\kappa_d(R)$. □

In particular, for dimension $d = 3$ this says that

$$\#C \cap \mathbb{Z}^3 \ll R^\epsilon(1 + r). \tag{2.9}$$

We can improve on Lemma 2.2 by slicing with well-chosen planes rather than vertical planes. More precisely, we have

LEMMA 2.3. *Let $C$ be a cap of size $r$ on the sphere $\{|\vec{x}| = R\} \subset \mathbb{R}^3$. Then for any $0 < \eta < 1/16$,*

$$\#C \cap \mathbb{Z}^3 \ll R^\epsilon \left(1 + r\left(\frac{r}{R}\right)^\eta\right). \tag{2.10}$$

*Proof.* It will involve several considerations.

(i) **Finding good slices.** We try to find an integer vector $\vec{a} \in \mathbb{Z}^d$ and use slices of the cap with the sections $\vec{a}.\vec{x} = k$. We consider a larger cap $C_1 = C(R\frac{\vec{a}}{|\vec{a}|}, R\theta_1)$ of radius $r_1 = R\theta_1$ around $R\frac{\vec{a}}{|\vec{a}|}$ which contains the original cap $C$. Thus we want the new cap angle $\theta_1$ to satisfy

$$\theta_1 = \theta + \left|\vec{\zeta} - \frac{\vec{a}}{|\vec{a}|}\right|. \tag{2.11}$$

To bound the number of lattice points in the new cap $C_1$, we exhaust them by the parallel sections $\vec{a}.\vec{x} = k$, which are orthogonal to the direction $\vec{a}$ of the new cap. The distance between adjacent sections is $1/|\vec{a}|$. The number of sections intersecting the cap $C_1$ is bounded by $|\vec{a}|$ times the height of the cap, which is $R - R\cos\theta_1 \approx R\theta_1^2$. Hence the number $\nu(C_1, \vec{a})$ of sections intersecting the cap is

$$\nu(C_1, \vec{a}) \ll 1 + R\theta_1^2 |\vec{a}| \tag{2.12}$$

and the analysis above shows that the number of lattice points in the cap $C_1$ is bounded by

$$\#C_1 \cap \mathbb{Z}^d \le \kappa_d(R) \cdot \nu(C_1, \vec{a}) \ll \kappa_d(R) \cdot (1 + R\theta_1^2 |\vec{a}|). \tag{2.13}$$

To gain over the linear bound (2.9) we need to find some $\delta > 0$ and a nonzero integer vector $\vec{a} \in \mathbb{Z}^d$ such that

$$R\theta_1^2 |\vec{a}| \ll r\theta^{2\delta} \tag{2.14}$$

that is

$$\theta + |\vec{\zeta} - \frac{\vec{a}}{|\vec{a}|}| \ll \frac{\theta^{\frac{1}{2}+\delta}}{|\vec{a}|^{\frac{1}{2}}}. \tag{2.15}$$

Setting

$$Q = \theta^{-1+2\delta} = \left(\frac{R}{r}\right)^{1-2\delta} \tag{2.16}$$

then (2.15) is implied by requiring both

$$|\vec{a}| \le Q \tag{2.17}$$

$$\left|\vec{\zeta} - \frac{\vec{a}}{|\vec{a}|}\right| \le \frac{1}{|\vec{a}|^{\frac{1}{2}} Q^{\frac{1}{2}+\gamma}} \tag{2.18}$$

where we have set

$$\gamma := \frac{2\delta}{1-2\delta}. \tag{2.19}$$

Finding $\vec{a} \in \mathbb{Z}^d$ as above is then our goal.

(ii) **Diophantine approximation.**                                    □

LEMMA 2.4. *Fix an integer $Q \geq 1$ and $0 < \eta < 1$. Let $\vec{\zeta} = (\zeta_1, \ldots, \zeta_d) \in [-1,1]^d$.*
*Then one of the following holds:*
   (1) *There is $Q \leq q \leq 2Q$ and $\vec{a} \in \mathbb{Z}^d$ such that*

$$\max_{1 \leq j \leq d} \left| \zeta_j - \frac{a_j}{q} \right| < \frac{\eta}{Q}. \tag{2.20}$$

   (2) *There are $\vec{b} \in \mathbb{Z}^d$, with $0 < \max |b_j| < 1/\eta$ with*

$$\left\| \sum_{j=1}^{d} b_j \zeta_j \right\| < \frac{c}{Q\eta^d} \tag{2.21}$$

*where we denote by $\|x\|$ the fractional part of $x$, or the distance of $x$ to the nearest
integer.*

*Proof.* Let $0 \leq \psi \leq 1$ be a smooth bump function on the torus $\mathbb{T}^d$, such that

   (1) $0 \leq \psi \leq 1$ for $\|x\| < \eta/2$
   (2) $\psi(x) = 0$ for $\|x\| > \eta$
   (3) $|\hat{\psi}(m)| \ll \eta^d e^{-\sqrt{\eta|m|}}$.

   If (2.20) fails, then

$$\max_{Q \leq q < 2Q, 1 \leq j \leq d} \|q\zeta_j\| \geq \eta$$

hence

$$\sum_{Q \leq q < 2Q} \psi(q\vec{\zeta}) = 0. \tag{2.22}$$

Expressing this in a Fourier series gives (writing $e(x) := e^{2\pi i x}$)

$$0 = Q\hat{\psi}(0) + \sum_{0 \neq b \in \mathbb{Z}^d} \hat{\psi}(b) \sum_{Q \leq q < 2Q} e(q\zeta \cdot b)$$

$$> cQ\eta^d - c \sum_{b \neq 0} \eta^d e^{-\sqrt{\eta|b|}} \left( |e(\zeta \cdot b) - 1| + \frac{1}{Q} \right)^{-1}$$

$$> cQ\eta^d \left( 1 - c \sum_{b \neq 0} e^{-\sqrt{\eta|b|}} \frac{1}{Q\|\zeta \cdot b\| + 1} \right)$$

$$> cQ\eta^d \left( \frac{1}{2} - c\eta^{-d} \max_{0 < |b| < c\eta^{-1}} \frac{1}{Q\|\zeta \cdot b\|} \right).$$

Hence $Q\|\zeta \cdot b\| < c\eta^{-d}$ for some nonzero $b \in \mathbb{Z}^d, |b| < c\eta^{-1}$.

LEMMA 2.5. *Let* $\zeta_1, \zeta_2 \in [-1, 1], 0 < \gamma < 1/15$, *and* $Q \gg_\gamma 1$ *an integer.*
  *Then there is an integer* $1 \le q \le Q$ *and* $\vec{a} \in \mathbb{Z}^2$ *so that*

$$\max_{j=1,2} \left| \zeta_j - \frac{a_j}{q} \right| \ll \frac{1}{q^{\frac{1}{2}} Q^{\frac{1}{2}+\gamma}}. \tag{2.23}$$

REMARK. Dirichlet's principle says that given $\vec{\zeta} \in \mathbb{R}^2$, and an integer $K \ge 1$, we can find $1 \le q \le K^2$ and $\vec{a} \in \mathbb{Z}^2$ so that

$$\max_{j=1,2} \left| \zeta_j - \frac{a_j}{q} \right| < \frac{1}{qK}. \tag{2.24}$$

Lemma 2.5 improves on this when $q$ is small.

*Proof.* Applying Lemma 2.4 with $\eta = Q^{-\gamma}$, either we have an integer $Q \le q \le 2Q$ with

$$\left| \zeta_j - \frac{a_j}{q} \right| < \frac{1}{Q^{1+\gamma}} \le \frac{\sqrt{2}}{q^{\frac{1}{2}} Q^{\frac{1}{2}+\gamma}}$$

which gives us what we need, or else the second option in the statement of the lemma occurs, that is there is some nonzero vector $\vec{b} \in \mathbb{Z}^2$ with $|b_1| \le |b_2| \le Q^\gamma$, and $a \in \mathbb{Z}$ so that

$$|b_1 \zeta_1 + b_2 \zeta_2 + a| < \frac{1}{Q^{1-2\gamma}} \tag{2.25}$$

that is

$$\left| \zeta_2 + \frac{b_1}{b_2} \zeta_1 + \frac{a}{b_2} \right| < \frac{1}{|b_1| Q^{1-2\gamma}}. \tag{2.26}$$

Now choose an integer $Q_1$ so that

$$2Q^{\frac{1+3\gamma}{2}} < Q_1 < \frac{1}{4} Q^{1-6\gamma} \tag{2.27}$$

which is possible if $0 < \gamma < 1/15$ and $Q \gg_\gamma 1$.
  Using Dirichlet's principle, there is some $1 \le q_1 \le Q_1$ and an integer $a' \in \mathbb{Z}$ so that

$$\left| \zeta_1 - \frac{a'}{q_1} \right| < \frac{1}{q_1 Q_1}. \tag{2.28}$$

Define $a_1, a_2 \in \mathbb{Z}$ by

$$a_1 = a'|b_2|, \quad -a_2 = \pm(b_1 a' + a q_1), \quad q = q_1 |b_2|. \tag{2.29}$$

We claim that these satisfy the statement of the Lemma. Indeed, by (2.28) we have

$$\left| \zeta_1 - \frac{a_1}{q} \right| = \left| \zeta_1 - \frac{a'}{q_1} \right| < \frac{1}{q_1 Q_1} \tag{2.30}$$

and due to (2.27) we have, since $q_1 = q/|b_2| \geq qQ^{-\gamma}$, that

$$\frac{1}{q_1 Q_1} < \frac{1}{2q_1 Q^{\frac{1}{2}+\frac{3\gamma}{2}}} < \frac{1}{2q_1^{\frac{1}{2}} Q^{\frac{1}{2}+\frac{3\gamma}{2}}} \leq \frac{1}{2q^{\frac{1}{2}} Q^{\frac{1}{2}+\gamma}} \tag{2.31}$$

giving $|\zeta_1 - \frac{a_1}{q}| < \frac{1}{2q^{\frac{1}{2}} Q^{\frac{1}{2}+\gamma}}$. Moreover using the small linear relation (2.26) between $\zeta_1$ and $\zeta_2$ and replacing $\zeta_1$ by $a_1/q = a'/q_1$ we find

$$\begin{aligned}
\left|\zeta_2 - \frac{a_2}{q}\right| &= \left|\zeta_2 + \frac{b_1}{b_2}\frac{a'}{q_1} + \frac{a}{b_2}\right| \\
&\leq \left|\zeta_2 + \frac{b_1}{b_2}\zeta_1 + \frac{a}{b_2}\right| + \frac{|b_1|}{|b_2|}\left|\zeta_1 - \frac{a'}{q_1}\right| \\
&< \frac{1}{|b_2|Q^{1-2\gamma}} + \frac{1}{q_1 Q_1}.
\end{aligned}$$

Now since $q_1 < Q_1 < \frac{1}{4}Q^{1-6\gamma}$ we have

$$\begin{aligned}
\frac{1}{|b_2|Q^{1-2\gamma}} &\leq \frac{1}{|b_2|^{\frac{1}{2}}Q^{1-2\gamma}} = \frac{q_1^{\frac{1}{2}}}{q^{\frac{1}{2}}Q^{1-2\gamma}} \\
&< \frac{1}{2q^{\frac{1}{2}}Q^{\frac{1}{2}+\gamma}}
\end{aligned} \tag{2.32}$$

and combining with (2.31) we get

$$\left|\zeta_2 - \frac{a_2}{q}\right| < \frac{1}{q^{\frac{1}{2}}Q^{\frac{1}{2}+\gamma}}$$

as claimed. $\qquad\square$

**2.6   Proof of Lemma 2.3.**   Assuming that $|\zeta_3| = \max|\zeta_j|$, we apply Lemma 2.5 to $\left(\frac{\zeta_1}{\zeta_3}, \frac{\zeta_2}{\zeta_3}\right)$ to find $1 \leq q \leq Q$ and $a_1, a_2 \in \mathbb{Z}$ so that

$$\max_{j=1,2}\left|\frac{\zeta_j}{\zeta_3} - \frac{a_j}{q}\right| < \frac{1}{q^{\frac{1}{2}}Q^{\frac{1}{2}+\gamma}}. \tag{2.33}$$

Set $\vec{a} = (a_1, a_2, q)$ then $|\vec{a}| \approx q$ and

$$\left|\vec{\zeta} - \zeta_3\frac{1}{q}\vec{a}\right| < \frac{1}{q^{\frac{1}{2}}Q^{\frac{1}{2}+\gamma}} \approx \frac{1}{|\vec{a}|^{\frac{1}{2}}Q^{\frac{1}{2}+\gamma}}. \tag{2.34}$$

Since for any pair of nonzero vectors $\vec{c}$, $\vec{d}$ we have by the triangle inequality

$$\left|\frac{\vec{c}}{|\vec{c}|} - \frac{\vec{d}}{|\vec{d}|}\right| \leq 2\frac{|\vec{c} - \vec{d}|}{|\vec{c}|} \tag{2.35}$$

and hence also

$$\left|\vec{\zeta} - \frac{\vec{a}}{|\vec{a}|}\right| \ll \frac{1}{|\vec{a}|^{\frac{1}{2}}Q^{\frac{1}{2}+\gamma}}.$$

Thus we have found $\vec{a}$ satisfying (2.17), (2.18), completing the proof of Lemma 2.3.

$\qquad\square$

# 3  The Fourier Transform of Surface-Carried Measures

Let $\Sigma \subset \mathbb{R}^3$ be a real analytic surface with non-vanishing curvature and $p \in \Sigma$. Applying a rigid motion, we may assume $p = 0$ and $\Sigma$ locally parametrized around $(0, 0, 0)$ by a map

$$(x_1, x_2) \mapsto \big(x_1, x_2, \phi(x_1, x_2)\big) \tag{3.1}$$

where $\phi$ is real-analytic on a neighborhood of $(0, 0)$ as has the form

$$\phi(x_1, x_2) = a_1 x_1^2 + a_2 x_2^2 + \sum_{\alpha + \beta \geq 3} a_{\alpha\beta}\, x_1^\alpha\, x_2^\beta \tag{3.2}$$

with

$$a_1 \neq 0, \quad a_2 \neq 0, \quad |a_{\alpha\beta}| < C^{\alpha+\beta} \ .$$

Distinguishing the case $a_1 a_2 > 0$ (positive curvature) and $a_1 a_2 < 0$ (negative curvature), we need to consider the two models

$$\phi(x_1, x_2) = x_1^2 + x_2^2 + \sum_{\alpha + \beta \geq 3} a_{\alpha\beta}\, x_1^\alpha\, x_2^\beta \tag{3.3}$$

and

$$\phi(x_1, x_2) = x_1^2 - x_2^2 + \sum_{\alpha + \beta \geq 3} a_{\alpha\beta}\, x_1^\alpha\, x_2^\beta. \tag{3.4}$$

Denote by $\sigma$ the surface measure of $\Sigma$. Let $\xi \in \mathbb{R}^3$ ($|\xi|$ large) and evaluate the Fourier transform

$$\int\limits_{\Sigma \text{ (local)}} e^{ix\xi}\sigma(dx) = \int e^{i\big(x_1\xi_1 + x_2\xi_2 + \phi(x_1,x_2)\xi_3\big)}\omega(x)\, dx_1\, dx_2 \tag{3.5}$$

where $\omega$ is some smooth function supported by a (small) neighborhood of $(0, 0)$.

The critical points of the phase function satisfy

$$\begin{cases} \xi_1 + \partial_1\phi(x)\xi_3 = \xi_1 + (2x_1 + \sum_{\alpha+\beta\geq 3} \alpha a_{\alpha\beta} x_1^{\alpha-1} x_2^\beta)\xi_3 = 0 \\ \xi_2 + \partial_2\phi(x)\xi_3 = \xi_2 + (2\epsilon x_2 + \sum_{\alpha+\beta\geq 3} \beta a_{\alpha\beta} x_1^\alpha x_1^{\beta-1})\xi_3 = 0 \end{cases} \tag{3.6}$$

where $\epsilon = \pm 1$ depending on whether we are in case (3.3) or (3.4).

It follows that in $\operatorname{supp}\omega$ there are no critical points unless

$$|\xi_1|, |\xi_2| < c|\xi_3| \tag{3.7}$$

($c$ a small constant, depending on $\operatorname{supp}\omega$).

If (3.7) there is a unique critical point

$$x = x(\xi) = \left(x_1\left(\frac{\xi_1}{\xi_3}, \frac{\xi_2}{\xi_3}\right), x_2\left(\frac{\xi_1}{\xi_3}, \frac{\xi_2}{\xi_3}\right)\right) \tag{3.8}$$

where

$$\begin{cases} 2x_1 + \sum_{\alpha+\beta\geq 3} \alpha a_{\alpha\beta} x_1^{\alpha-1} x_2^\beta = -\frac{\xi_1}{\xi_3} \\ 2\epsilon x_2 + \sum_{\alpha+\beta\geq 3} \beta a_{\alpha\beta} x_1^\alpha x_2^{\beta-1} = -\frac{\xi_2}{\xi_3} \end{cases} \tag{3.9}$$

and $|a_{\alpha\beta}| < C^{\alpha+\beta}$.

From the stationary phase formula (see [GS77], Ch. 1)

$$(3.5) = \frac{2\pi}{|\xi_3|} e^{\frac{\pi i}{4}\operatorname{sign} H(x(\xi))} \frac{e^{i\psi(\xi)}}{|\det H(x(\xi))|^{1/2}} \omega(x(\xi)) + 0\left(\frac{1}{|\xi|^2}\right) \tag{3.10}$$

where $H(x)$ is the Hessian of $\phi$ at $x$, $\operatorname{sign} H$ is the signature of $H$ and

$$\psi(\xi) = x_1(\xi)\xi_1 + x_2(\xi)\xi_2 + \phi(x_1(\xi), x_2(\xi))\xi_3$$

$$\overset{(3.6)}{=} -\xi_3\{x_1(\xi)\partial_1\phi(x(\xi)) + x_2(\xi)\partial_2\phi(x(\xi)) - \phi(x_1(\xi), x_2(\xi))\}. \tag{3.11}$$

By (3.3), (3.4)

$$H(x) = \begin{bmatrix} 2 + \sum_{\substack{\alpha+\beta\geq 3 \\ \alpha\geq 2}} \alpha(\alpha-1)\, a_{\alpha\beta}\, x_1^{\alpha-2}\, x_2^\beta & \sum_{\substack{\alpha+\beta\geq 3 \\ \alpha\geq 1,\beta\geq 1}} \alpha\beta a_{\alpha\beta}\, x_1^{\alpha-1}\, x_2^{\beta-1} \\[2mm] \sum_{\substack{\alpha+\beta\geq 3 \\ \alpha\geq 1,\beta\geq 1}} \alpha\beta\, a_{\alpha\beta}\, x_1^{\alpha-1}\, x_2^{\beta-1} & 2\epsilon + \sum_{\substack{\alpha+\beta\geq 3 \\ \beta\geq 2}} \beta(\beta-1)\, a_{\alpha\beta}\, x_1^\alpha\, x_2^{\beta-2} \end{bmatrix} \tag{3.12}$$

and hence $\operatorname{sign} H = 2$ (resp. 0) for positive (resp. negative) curvature.

As will be clear soon, the error term $0(|\xi|^{-2})$ will be harmless in our analysis in the restriction problem for $\mathbb{T}^3$-eigenfunctions. The relevant contribution will be

$$\frac{e^{i\psi(\xi)}}{|\xi|} \tag{3.13}$$

coming from the main term. It turns out that the decay factor $\frac{1}{|\xi|}$ is barely insufficient to ignore the oscillatory factor $e^{i\psi(\xi)}$. In order to exploit this factor, a more careful analysis of the phase function $\psi(\xi)$ is necessary.

Returning to (3.9), (3.11), we obtain by the implicit function theorem (recalling (3.7)).

$$\begin{cases} x_1(\xi) = -\frac{\xi_1}{2\xi_3} + \sum_{\alpha+\beta\geq 2} b_{\alpha\beta}\left(\frac{\xi_1}{\xi_3}\right)^\alpha \left(\frac{\xi_2}{\xi_3}\right)^\beta \\ x_2(\xi) = -\epsilon\frac{\xi_2}{2\xi_3} + \sum_{\alpha+\beta\geq 2} c_{\alpha\beta}\left(\frac{\xi_1}{\xi_3}\right)^\alpha \left(\frac{\xi_2}{\xi_3}\right)^\beta \end{cases}$$

and

$$\psi(\xi) = -\frac{1}{4}\left(\frac{\xi_1^2}{\xi_3} + \epsilon\frac{\xi_2^2}{\xi_3}\right) + \sum_{\alpha+\beta\geq 3} d_{\alpha\beta}\, \xi_1^\alpha\, \xi_2^\beta\, \xi_3^{1-\alpha-\beta} \tag{3.14}$$

$(|b_{\alpha,\beta}|, |c_{\alpha\beta}|.|d_{\alpha\beta}| < C^{\alpha+\beta})$.

Thus $\psi(\xi)$ is homogeneous of degree one and hence $\nabla\psi(\xi)$ is radially constant and $D^2\psi(\lambda\xi) = \frac{1}{\lambda}D^2\psi(\xi)$. The self-adjoint matrix $D^2\psi(\xi), \xi \neq 0$, has $\xi$ as eigenvector with eigenvalue 0.

From (3.14)

$$D^2\psi(\xi) = \begin{pmatrix} -\frac{1}{2\xi_3} & 0 & 0 \\ 0 & -\frac{\epsilon}{2\xi_3} & 0 \\ 0 & 0 & 0 \end{pmatrix} + 0\Big(\frac{|\xi_1| + |\xi_2|}{\xi_3^2}\Big) \tag{3.15}$$

and by (3.7), we conclude that the other two eigenvalues of $D^2\psi(\xi)$ are of size $\sim \frac{1}{|\xi|}$ with same or opposite sign depending on $\epsilon = 1, -1$.

Hence

$$D^2\psi(\xi) = \frac{1}{|\xi|}P_{\xi^\perp}AP_{\xi^\perp} \tag{3.16}$$

where $A$ is a self-adjoint operator (depending on $\frac{\xi}{|\xi|}$), acting on $\xi^\perp$ and with eigenvalues bounded from above and below (with same sign for $\epsilon = 1$ and opposite sign for $\epsilon = -1$).

## 4 Spherical Restriction of the Phase Function

Let $\psi(\xi)$ be the phase function obtained in Sect. 3 and

$$S = S^2 = \{x \in \mathbb{R}^3, |x| = 1\}.$$

The domain of definition of $\psi$ is a cone $Z = \{|\xi_1|, |\xi_2| < c|\xi_3|\}$, with $c > 0$ a small constant, and $\psi$ is real analytic on $Z$.

Subcones $Z' = \{|\xi_1|, |\xi_2| < c'|\xi_3|\} \subset Z, c' < c$, will also be denoted by $Z$. We will need a normal form analysis of the function $\psi(x - y)$ with $x, y$ restricted to $S$.

LEMMA 4.1. *Let $p : O \overset{open}{\subset} \mathbb{R}^2 \to C$ be a real analytic parametrization of a cap $C \subset S$ such that $C \cap (\xi + \bar{Z})$ is connected for all $\xi \in C$.*

*Let $a, b \in O, a \neq b$ such that $p(a) - p(b) \in Z$. There are real analytic coordinate changes $\alpha$ (resp. $\beta$) on a neighborhood of $a$ (resp. $b$) such that*

$$\psi\big(p\big(a + \alpha(x)\big) - p(b + \beta(y))\big) = f(x) + g(y) + x_1y_1 + x_2y_2 + h(x, y) \tag{4.1}$$

*with $f, g, h$ real analytic, $h(x, y) = 0(|x|^2|y|^2)$ and $h \neq 0$.*

*Proof.* (i) Letting $\eta = p(a) - p(b), \bar{\eta} = \frac{\eta}{|\eta|}$, it follows from (3.16) and curvature that the quadratic form

$$D^2\psi(\eta) = \frac{1}{|\eta|}P_{\eta^\perp}A_{\bar{\eta}}P_{\eta^\perp}$$

is non-degenerate on $\big(T_a - p(a)\big) \times \big(T_b - p(b)\big)$ where $T_a$ (resp. $T_b$) is the tangent space at $p(a) \in S$ (resp. $p(b) \in S$), as in Fig. 1.

$C$
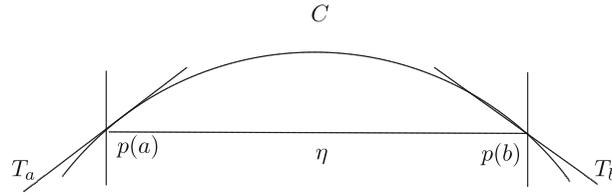
Figure 1: Tangent spaces in a cap

$V$

$w_0 + Z$

$w_1 + Z$

$w_1$

$p(b) = w_0$
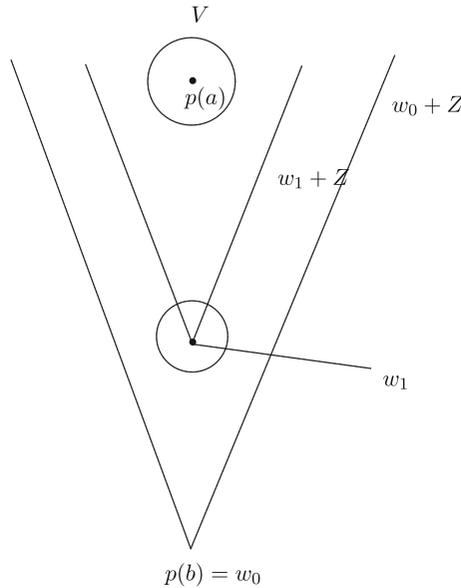
Figure 2: A neighborhoood of $p(a)$

Performing coordinate changes $\alpha, \beta$ in $x, y$ separately, we can therefore obtain the form (4.1) with $h(x, y) = 0(|x|^2 |y|^2)$. It remains to show that $h$ does not vanish identically.

(ii) Assume that $h = 0$. Then

$$\psi\big(p\big(a + \alpha(x)\big) - p\big(b + \beta(y)\big)\big) = f(x) + g(y) + x_1 y_1 + x_2 y_2 \tag{4.2}$$

for $x, y$ in a neighborhood of $(0, 0) \in \mathbb{R}^2$.

Define $f_w(v) = \psi(v - w)$ where $w \in S$ is in a neighborhood $W$ of $p(b)$ and $v \in S \cap (w + Z)$. It follows from (4.2) that there is a neighborhood $V$ of $p(a)$ in $S$ (Fig. 2) such that

$$\dim[f_w|_V; w \in W] \le 4. \tag{4.3}$$

Take $\delta_0 \gg \delta_1 \gg \cdots \gg \delta_4$ and points $p(b) = w_0, w_1, \ldots, w_4 \in S$ in $W$ satisfying

$$B(w_{i+1}, \delta_{i+1}) \subset B(w_i, \delta_i) \cap (w_i + Z).$$
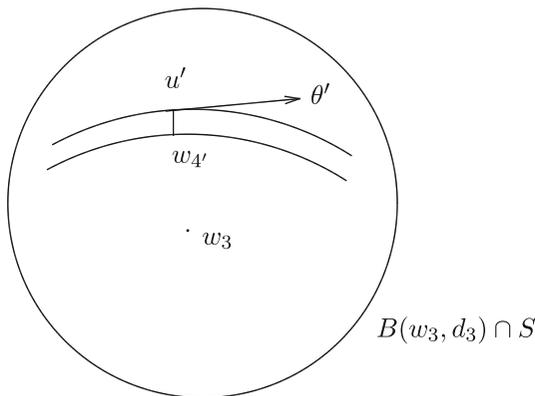
Figure 3: The negative definite case

From (4.3), we may assume $f_{w_4}|_V$ a linear combination of $f_{w_i}|_V$ $(0 \leq i \leq 3)$. Hence, invoking real analyticity, it follows that $f_{w_4}$ is a linear combination of $f_{w_i}(0 \leq i \leq 3)$ on $\bigcap_{i=0}^4 (w_i + Z) \cap S$. Since the functions $f_{w_i}(0 \leq i \leq 3)$ are smooth on $B(w_4, \delta_4) \cap S \subset \bigcap_{i=0}^3 (w_i + Z)$, it follows in particular that $f_{w_4}$ is smooth on $B(w_4, \delta_4) \cap (w_4 + \bar{Z}) \cap S$. Hence, taking $u \in B(w_4, \delta_4) \cap (w_4 + Z) \cap S$, it follows that

$$D^2 \psi(u - w_4) = \frac{1}{|\zeta|} P_{\zeta^\perp} A_{\frac{\zeta}{|\zeta|}} P_{\zeta^\perp}, \quad \zeta = u - w_4$$

restricted to $T_u - u$, is uniformly bounded for $u \in B(w_4, \delta_4) \cap (w_4 + Z) \cap S$.

Thus for $\zeta$ as above and $\theta, \xi \in T_u - u$, $|\theta| = 1 = |\xi|$,

$$\langle A P_{\zeta^\perp} \theta, \xi \rangle = 0(|\zeta|) \tag{4.4}$$

where $A = A_{\frac{\zeta}{|\zeta|}}$.

We show that this is not the case.

If $A$ is positive definite, take $\theta = \xi \in \zeta^\perp \cap (T_u - u)$, $|\theta| = 1$. Hence, from (4.4),

$$1 \sim \langle A\theta, \theta \rangle = 0(|u - w_4|). \tag{4.5}$$

Letting $u \to w_4$, we obtain a contradiction.

If $A$ is negative definite, proceed as follows.

Fix $\zeta = u - w_4$ and let $w_4'$ vary in $B(w_3, \delta_3) \cap S$ such that $u' = w_4' + \zeta \in B(w_3, \delta_3) \cap S$. Hence $w_4'$ varies over an arc of size $\sim \delta_3$ (see Fig. 3). Let

$$\theta' \in \zeta^\perp \cap (T_{u'} - u') = \zeta^\perp \cap (T_{w_4'} - w_4'), \quad |\theta'| = 1.$$

From (4.4)

$$\langle A\theta', \theta' \rangle = 0(|\zeta|) \tag{4.6}$$

where $A$ does not depend on $w_4'$ and $\theta'$ also varies over an arc of size $\sim \delta_3$. Thus the left side of (4.6) can be made at least $\sim \delta_3$, independently of $|\zeta|$, a contradiction.

This proves Lemma 4.1.                                                      $\square$

LEMMA 4.2. *In the situation of Lemma* 4.1, *we may choose* $a, b \in O$, $p(a) - p(b) \in Z$ *such that in* (4.1) *the function*

$$h(x,y) = x_1^2 Q_{11}(y) + x_1 x_2 Q_{12}(y) + x_2^2 Q_{22}(y) + 0(|x|^3|y|^2 + |x|^2|y|^3) \qquad (4.7)$$

*where the* $Q_{ij}(y)$ *are quadratic forms, not all zero.*

*Proof.* Start with (4.1) with $h(x,y) = 0(|x|^2|y|^2), h \neq 0$. Taking a sufficiently small $\delta > 0$, it follows from the mean value theorem that on $B(0,\delta) \times B(0,\delta)$, $B(0,\delta) \subset \mathbb{R}^2$

$$|\partial_x^2 \partial_y h| \leq \delta \big( \max_{B(0,\delta)} |\partial_x^2 \partial_y^2 h| \big), \qquad (4.8)$$

since $\partial_x^2 \partial_y h|_{y=0} = 0$.

Similarly for $\partial_x \partial_y^2 h$.

It follows that there are $\bar{x}, \bar{y} \in B(0,\delta) \cap \mathbb{R}^2$ such that

$$\|(\partial_x^2 \theta_y h)(\bar{x}, \bar{y})\| + \|(\partial_x \partial_y^2 h)(\bar{x}, \bar{y})\| < \delta \|(\partial_x^2 \partial_y^2 h)(\bar{x}, \bar{y})\| < 1. \qquad (4.9)$$

Setting $x = \bar{x} + \Delta x, y = \bar{y} + \Delta y$ in (4.1), we obtain after a linear coordinate change in $\Delta y$

$$\psi\big(p\big(a + \alpha(\bar{x} + \Delta x)\big) - p\big(b + \beta(\bar{y} + \Delta y)\big)\big)$$
$$= \bar{f}(\Delta x) + \bar{g}(\Delta y) + (\Delta x)_1 (\Delta y)_1 + (\Delta x)_2 (\Delta y)_2$$
$$+ \sum_{i,j,k=1,2} c_{ijk}(\partial_{x_i x_j} \partial_{y_k} h)(\bar{x}, \bar{y})(\Delta x)_i (\Delta x)_j (\Delta y)_k \qquad (4.10)$$

$$+ \sum_{i,j,k=1,2} c_{ijk}(\partial_{x_i} \partial_{y_j y_k} h)(\bar{x}, \bar{y})(\Delta x)_i (\Delta y)_j (\Delta y)_k \qquad (4.11)$$

$$+ \sum_{i,j,k,\ell=1,2} c_{ijk\ell}(\partial_{x_i x_j} \partial_{y_k y_\ell} h)(\bar{x}, \bar{y})(\Delta x)_i (\Delta x)_j (\Delta y)_h (\Delta y)_\ell \qquad (4.12)$$

$$+0(|\Delta x|^3|\Delta y| + |\Delta x| \ |\Delta y|^3) \qquad (4.13)$$
$$+0(|\Delta x|^3|\Delta y|^2 + |\Delta x|^2|\Delta y|^3)$$

where (4.10)–(4.12) satisfy (4.9).

We eliminate the $0(|\Delta x|^2|\Delta y|)$-terms in (4.10), (4.13) by a coordinate change in $\Delta x$ and then the $0(|\Delta x| \ |\Delta y|^2)$-terms by a coordinate change in $\Delta y$. Since the new quartic terms introduced by these coordinate changes (in fact only the first) have coefficients at most

$$0\big(\|(\partial_x^2 \partial_y h)(\bar{x}, \bar{y})\|.\|(\partial_x \partial_y^2 h)(\bar{x}, \bar{y})\|\big)$$
$$< \delta \|\partial_x^2 \partial_y^2 h(\bar{x}, \bar{y})\|$$

by (4.9), the resulting expression will clearly still have a nonvanishing bi-quadratic term. This proves Lemma 4.2. □

Denoting $F_{a,b}(x,y) = \psi\big(p(a+\alpha(x)) - p(b+\beta(y))\big)$ with $h(x,y)$ in (4.1) satisfying Lemma 4.2, it follows that the Wronskian

$$
\max_{i,j,k,\ell=1,2} W_{i,j,k,\ell}(F_{a,b})(0,0) \equiv \max_{i,j,k,\ell=1,2}
\begin{vmatrix}
\partial_{x_1}\partial_{y_1} F & \partial_{x_1}\partial_{y_2} F & \partial_{x_1}\partial_{y_k y_\ell} F \\
\partial_{x_2}\partial_{y_1} F & \partial_{x_2}\partial_{y_2} F & \partial_{x_2}\partial_{y_k y_\ell} F \\
\partial_{x_i x_j}\partial_{y_1} F & \partial_{x_i x_j}\partial_{y_2} F & \partial_{x_i x_j}\partial_{y_k y_\ell} F
\end{vmatrix}(0,0) \neq 0.
$$

(4.14)

Note that property (4.14) does not depend on the parametrization. Thus

$$
\max W_{ijk\ell}(a,b) = \max W_{i,j,k,\ell}\big(\psi\big(p(x) - p(y)\big)\big)(a,b) \neq 0. \tag{4.15}
$$

Invoking real analyticity, we obtain

LEMMA 4.3. *With previous notations, the set*

$$
\left\{ (x,y) \in O \times O; p(x) - p(y) \in Z; \max_{i,j,k,\ell} W_{ijk\ell}(x,y) = 0 \right\}
$$

*is at most a 3-dim submanifold.*

Also, for $\delta_1 > \delta > 0$ small enough and considering a partition of $O$ in $\delta$-boxes $Q_\alpha$, we have

$$
\begin{aligned}
\#\mathcal{W}_{\delta,\delta_1} = \#&\left\{ (\alpha,\beta); (p(Q_\alpha) - p(Q_\beta)) \cap Z \neq \phi \text{ and } \max_{i,j,k,\ell} \min_{Q_\alpha \times Q_\beta} |W_{ijk\ell}(x,y)| < \delta_1 \right\} \\
&< \delta^{-4}\delta_1^{c_1}
\end{aligned}
$$

(4.16)

*(for some constant $c_1$ independent of $\delta_1$).*

Fix $\alpha \neq \beta$ such that $p(Q_\alpha) - p(Q_\beta) \subset Z$ and $(\alpha,\beta)$ not in the exceptional set $\mathcal{W} = \mathcal{W}_{\delta,\delta_1}$. Let $Q_\alpha = a_\alpha + U_\alpha, Q_\beta = a_\beta + U_\beta$ where $Q_\alpha, Q_\beta \subset O$ and $U_\alpha, U_\beta$ are $\delta$-neighborhoods of $(0,0)$.

Appropriate coordinate changes in $x, y$ permit to bring $\psi\big(p(a_\alpha + x) - p(a_\beta + y)\big)$ in the form

$$
\begin{aligned}
&f(x) + g(y) + x_1 y_1 + x_2 y_2 + x_1^2 Q_{11}(y) + x_1 x_2 Q_{12}(y) + x_2^2 Q_{22}(y) \\
&\quad + O(|x|^2 |y|^2(|x| + |y|))
\end{aligned}
$$

(4.17)

with

$$
\max_{i,j=1,2} \|Q_{ij}\| > \delta_1. \tag{4.18}
$$

Next, we show

LEMMA 4.4. *Further linear coordinate changes in $x$ and $y$ provide an expression of the form*

$$f(x) + g(y) + x_1 y_1 + x_2 y_2 + q x_1^2 y_1^2$$
$$+ O\big((|x_2||x| + |y_2||y|)(|x| + |y|)^2 + |x|^2|y|^2(|x| + |y|)\big) \qquad (4.19)$$

*with $|q| \gtrsim \delta_1$.*

*Proof.* With $a$ a parameter to be specified, make a linear transformation

$$x \mapsto (x_1, x_2 + ax_1) \qquad y \to (y_1 - ay_2, y_2)$$

preserving the quadratic part of (4.17). We obtain

$$f_1(x) + g_1(y) + x_1 y_1 + x_2 y_2 + x_1^2 Q_{11}(y_1 - ay_2, y_2)$$
$$+ x_1(x_2 + ax_1) Q_{12}(y_1 - ay_2, y_2) + (x_2 + ax_1)^2 Q_{22}(y_1 - ay_2, y_2)$$
$$+ O\big(|x|^2|y|^2(|x| + |y|)\big)$$

with bi-quadratic part

$$x_1^2 [Q'_{11}(y) + a Q'_{12}(y) + a^2 Q'_{22}(y)] + 0(|x_2||x||y|^2) \qquad (4.20)$$

where

$$Q'_{ij}(y) = Q_{ij}(y_1 - ay_2, y_2)$$

satisfies, by (4.18)

$$\max_{i,j} \|Q'_{ij}\| > \delta_1. \qquad (4.21)$$

Clearly there is some $a = O(1)$ such that

$$\|Q'\| = \|Q'_{11} + a Q'_{12} + a^2 Q'_{22}\| \gtrsim \delta_1. \qquad (4.22)$$

Thus after this first linear transformation, we get

$$f_1(x) + g_1(y) + x_1 y_1 + x_2 y_2 + x_1^2 Q'(y_1, y_2)$$
$$+ O\big(|x_2||x||y|^2 + |x|^2|y|^2(|x| + |y|)\big) \qquad (4.23)$$

and

$$Q'(y_1, y_2) = q_{11} y_1^2 + q_{12} y_1 y_2 + q_{22} y_2^2$$

satisfying

$$\max_{i,j} |q_{ij}| \gtrsim \delta_1. \qquad (4.24)$$

Next, make a second transformation

$$x \mapsto (x_1 - bx_2, x_2) \qquad y \mapsto (y_1, y_2 + by_1)$$

with $b = O(1)$, converting (4.23) to

$$
\begin{aligned}
f_2(x) &+ g_2(y) + x_1 y_1 + x_2 y_2 \\
&+ (x_1 - bx_2)^2 [q_{11} y_1^2 + q_{12} y_1 (y_2 + by_1) + q_{22}(y_2 + by_1)^2] \\
&+ O\left(|x_2|\,|x|\,|y|^2 + |x|^2 |y|^2 (|x| + |y|)\right) \\
= f_2(x) &+ g_2(y) + x_1 y_1 + x_2 y_2 + x_1^2 y_1^2 (q_{11} + b q_{12} + b^2 q_{22}) \\
&+ O\left(|x_2|\,|x|\,|y|^2 + |y_2|\,|y|\,|x|^2 + |x|^2 |y|^2 (|x| + |y|)\right).
\end{aligned}
\tag{4.25}
$$

By (4.24), we can choose $b$ such that

$$
|q| = |q_{11} + b q_{12} + b^2 q_{22}| \gtrsim \delta_1.
$$

This proves Lemma 4.4.                                                                 $\square$

## 5  Estimation of Certain Oscillatory Sums

Let $E = R^2 \in \mathbb{Z}_+$ be the eigenvalue.

In the preceding Sect. 3, we take $\delta = R^\varepsilon$ with $\varepsilon > 0$ a small constant and $\delta_1 = \sqrt{\delta}$. Our purpose in this section is to establish nontrivial bounds on sums of the form

$$
\sum_{x \in X, y \in Y} e^{iR[\psi(p(a_\alpha + x) - p(a_\beta + y))]}
\tag{5.1}
$$

where $X \subset U_\alpha, Y \subset U_\beta$ are discrete sets of $\frac{1}{\sqrt{R}}$-separated points (recall that $U_\alpha, U_\beta$ are $\delta$-neighborhoods of $(0,0)$). The sets $X, Y$ will correspond to diffeomorphic images of subsets of $\mathcal{E} = RS^2 \cap \mathbb{Z}^3$ as we will explain in Sect. 7.

Our aim is to prove an estimate

$$
|(5.1)| < R^{2-\kappa}
\tag{5.2}
$$

for some $\kappa > 0$ (independent of $R$).

The bound (5.2) will be derived from the following 1-dimensional inequality.

LEMMA 5.1. *Assume $S, T \subset [0, R^{-\frac{1}{5}}]$ arbitrary discrete sets of $\frac{1}{\sqrt{R}}$-separated points and $0 < |q| < 0(1)$. Then*

$$
\left| \sum_{s \in S, t \in T} e^{iR(st + qs^2 t^2)} \right| < R^{\frac{3}{5} - \kappa_1} |q|^{-1}
\tag{5.3}
$$

*for some constant $\kappa_1 > 0$.*

Lemma 5.1 will be proven in Sect. 6. In this section, we derive (5.2) from (5.3). According to Lemma 4.4, we may assume for $x \in U_\alpha, y \in U_\beta$

$$
\begin{aligned}
\psi\big(p(a_\alpha + x) - p(a_\beta + y)\big) &= f(x) + g(y) + x_1 y_1 + x_2 y_2 + q x_1^2 y_1^2 \\
&\quad + O\big((|x_2|\,|x| + |y_2|\,|y|)(|x| + |y|)^2\big) + O\big(|x|^2 |y|^2 (|x| + |y|)\big)
\end{aligned}
\tag{5.4}
$$

where $|q| > \delta$.

In order to reduce the problem to a 1-dimensional setting, a further restriction of the range of the $x, y$-variables will be performed.

Let $\bar{x} \in U_\alpha, \bar{y} \in U_\beta$ and $x = \bar{x} + x', y = \bar{y} + y'$ with $x', y'$ suitably restricted. Write

$$\psi\big(p(a_\alpha + \bar{x} + x') - p(a_\beta + \bar{y} + y')\big)$$

$$= \psi\big(p(a_\alpha + x') - p(a_\beta + y')\big) + \sum_{i=1}^{2} \bar{x}_i A_i(\bar{x}, \bar{y}, x', y') + \sum_{j=1}^{2} \bar{y}_j B_j(\bar{x}, \bar{y}, x', y')$$

$$\overset{5.4}{=} f(x') + g(y') + x'.y'$$
$$+ q(x_1')^2 (y_1')^2 + O\big((|x_2'|\,|x'| + |y_2'|\,|y'|)(|x'|^2 + |y'|^2) + |x'|^2|y'|^2(|x'| + |y'|)\big)$$

$$+ \sum_{i=1}^{2} \bar{x}_i A_i(\bar{x}, \bar{y}, x', y') + \sum_{j=1}^{2} \bar{y}_j B_j(\bar{x}, \bar{y}, x', y'). \tag{5.5}$$

Perform coordinate changes in $x', y'$ separately (as described in Lemma 4.1)

$$\begin{cases} x' = \zeta_{\bar{x},\bar{y}}^{(1)}(x'') \\ y' = \zeta_{\bar{x},\bar{y}}^{(2)}(y'') \end{cases} \tag{5.6}$$

in order to bring (5.5) in the form

$$\psi\big(p(a_\alpha + \bar{x} + \zeta_{\bar{x},\bar{y}}^{(1)}(x'')) - p(a_\beta + \bar{y} + \zeta_{\bar{x},\bar{y}}^{(2)}(y''))\big) = f_1(x'') + g_1(y'') + x''.y'' + h(x'', y'') \tag{5.7}$$

where

$$h(x'', y'') = O(|x''|^2 |y''|^2).$$

Clearly $\zeta_{\bar{x},\bar{y}}^{(1)}, \zeta_{\bar{x},\bar{y}}^{(2)}$ depend real-analytically on $\bar{x}, \bar{y}$.

Also, since $|\bar{x}|, |\bar{y}| < \delta$

$$\begin{cases} \zeta_{\bar{x},\bar{y}}^{(1)}(x'') = x'' + O\big((|\bar{x}| + |\bar{y}|)|x''|\big) \\ \zeta_{\bar{x},\bar{y}}^{(2)}(y'') = y'' + O\big((|\bar{x}| + |\bar{y}|)|y''|\big) \end{cases} \tag{5.8}$$

are $\delta$-perturbations of identity.

Returning to (5.5), it follows that

$$h(x'', y'') = q(x_1'')^2 (y_1'')^2 + O\big((|x_2''|\,|x''| + |y_2''|\,|y''|)(|x''| + |y''|)^2 + |x''|^2|y''|^2(|x''| + |y''|)\big)$$

$$+ O\big((|\bar{x}| + |\bar{y}|)|x''|^2|y''|^2\big)$$

$$= q''(x_1'')^2 (y_1'')^2 + O\big((|x_2''|\,|x''| + |y_2''|\,|y''|)(|x''|^2 + |y''|^2)\big)$$

$$+ O\big(|x''|^2|y''|^2(|x''| + |y''|)\big) \tag{5.9}$$

where $q'' = q + O(\delta)$, hence $|q''| > \frac{1}{2}|q| \gtrsim \delta_1$.

Thus

$$(5.7) = f_1(x'') + g_1(y'') + x_1'' y_1'' + x_2'' y_2'' + q''(x_1'')^2(y_1'')^2$$

$$+ O\big((|x_2''| \, |x''| + |y_2''| \, |y''|)(|x''|^2 + |y''|^2)\big)$$

$$+ O\big((|x''| + |y''|)^5\big). \tag{5.10}$$

Fix a small parameter $\tau > 0$ and denote

$$B = [0, R^{-\frac{1}{5}-\tau}] \times [0, R^{-\frac{1}{2}-\tau}]. \tag{5.11}$$

If we restrict $x'' \in B, y'' \in B$, clearly

$$(5.10) = f_1(x'') + g_1(y'') + x_1'' y_1'' + q''(x_1'')^2(y_1'')^2$$

$$+ O(R^{-1-2\tau} + R^{-\frac{3}{5}-\frac{1}{2}} + R^{-1-5\tau}). \tag{5.12}$$

Hence, returning to (5.1)

$$\left| \sum_{\substack{x'',y'' \in B \\ \zeta_{\bar{x},\bar{y}}^{(1)}(x'') \in X - \bar{x} \\ \zeta_{\bar{x},\bar{y}}^{(2)}(y'') \in Y - \bar{y}}} e^{R\psi\big(p(a_\alpha + \bar{x} + \zeta_{\bar{x}\bar{y}}^{(1)}(x'')) - p(a_\beta + \bar{y} + \zeta_{\bar{x}\bar{y}}^{(2)}(y''))\big)} \right|$$

$$\leq \left| \sum_{\substack{x'',y'' \in B \\ \zeta_{\bar{x},\bar{y}}^{(1)}(x'') \in X - \bar{x} \\ \zeta_{\bar{x},\bar{y}}^{(2)}(y'') \in Y - \bar{y}}} c(x'') d(y'') c^{iR[x_1'' y_1'' + q''((x_1'')^2(y_1'')^2)]} \right| \tag{5.13}$$

$$+ O\big(R^{-2\tau}|X \cap [\zeta_{\bar{x},\bar{y}}^{(1)}(B) + \bar{x}]| \cdot |Y \cap [\zeta_{\bar{x},\bar{y}}^{(2)}(B) + \bar{y}]|\big) \tag{5.14}$$

with $|c(x'')| = |d(y'')| = 1$.

Recall that $X, Y$ consist of $\frac{1}{\sqrt{R}}$-separated points. Hence also the elements of $(\zeta_{\bar{x},\bar{y}}^{(1)})^{-1}(X - \bar{x})$ and $(\zeta_{\bar{x},\bar{y}}^{(2)})^{-1}(Y - \bar{y})$ are $\sim \frac{1}{\sqrt{R}}$-separated. From the definition (5.11) of $B$, it follows that

$$S = \pi_1[B \cap (\zeta_{\bar{x},\bar{y}}^{(1)})^{-1}(X - \bar{x})]$$

$$T = \pi_1[B \cap (\zeta_{\bar{x},\bar{y}}^{(2)})^{-1}(Y - \bar{y})]$$

are $\sim \frac{1}{\sqrt{R}}$ separated.

Assuming a general estimate (5.3) ($\kappa_1 > 0$ some fixed constant) at our disposal, we can therefore conclude that

$$|(5.13)| < R^{-\kappa_1 + \frac{3}{5}} \frac{1}{|q''|} < R^{-\frac{1}{2}\kappa_1 + \frac{3}{5}}. \tag{5.15}$$

In conclusion, we obtain from (5.13)–(5.15)

$$\left| \sum_{\substack{x \in X \cap (\bar{x} + \zeta^{(1)}_{\bar{x},\bar{y}}(B)) \\ y \in Y \cap (\bar{y} + \zeta^{(2)}_{\bar{x},\bar{y}}(B))}} e^{iR\psi(p(a_\alpha + x) - p(a_\beta + y))} \right|$$
$$\lesssim R^{-2\tau} |X \cap (\bar{x} + \zeta^{(1)}_{\bar{x}\bar{y}}(B))| . |Y \cap (\bar{y} + \zeta^{(2)}_{\bar{x}\bar{y}}(B))| + R^{\frac{3}{5} - \frac{1}{2}\kappa_1}. \qquad (5.16)$$

Recall that $\bar{x} \in U_\alpha, \bar{y} \in U_\beta$ were arbitrarily chosen.

Integration of (5.16) in $\bar{x} \in U_\alpha, \bar{y} \in U_\beta$ gives

$$\sum_{x \in X, y \in Y} e^{iR\psi(p(a_\alpha + x) - p(a_\beta + y))} \left\{ \iint_{U_\alpha \times U_\beta} [1_{\bar{x} + \zeta^{(1)}_{\bar{x}\bar{y}}(B)}(x) . 1_{\bar{y} + \zeta^{(2)}_{\bar{x}\bar{y}}(B)}(y)] d\bar{x} d\bar{y} \right\}$$
$$\lesssim R^{-2\tau} \sum_{x \in X, y \in Y} \left\{ \iint_{U_\alpha \times U_\beta} [1_{\bar{x} + \zeta^{(1)}_{\bar{x}\bar{y}}(B)}(x) 1_{\bar{y} + \zeta^{(2)}_{\bar{x}\bar{y}}(B)}(y)] d\bar{x} d\bar{y} \right\} + R^{\frac{3}{5} - \frac{1}{2}\kappa_1}. \qquad (5.17)$$

Next, we analyze the expression { }.

For fixed $x, y$, consider the equations

$$\begin{cases} x = \bar{x} + \zeta^{(1)}_{\bar{x},\bar{y}}(x'') \\ y = \bar{y} + \zeta^{(2)}_{\bar{x},\bar{y}}(y'') \end{cases} \qquad (5.18)$$

with $x'', y'' \in B$. Note that by (5.8)

$$|\partial_{\bar{x}} \zeta^{(1)}_{\bar{x}\bar{y}}(x'')| + |\partial_{\bar{y}} \partial^{(1)}_{\bar{x}\bar{y}}(x'')| + |\partial_{\bar{x}} \zeta^{(2)}_{\bar{x}\bar{y}}(y'')| + |\partial_{\bar{y}} \zeta^{(2)}_{\bar{x}\bar{y}}(y'')| < O(|x''| + |y''|) < R^{-\frac{1}{5}}.$$

Hence, by the implicit function theorem, (5.18) may be rewritten as

$$(\bar{x}, \bar{y}) = \Omega_{x,y}(x'', y'') \qquad (5.19)$$

where $\Omega_{x,y}$ is a diffeomorphism from $B \times B$ to $\Omega_{x,y}(B \times B) \subset U_\alpha \times U_\beta$ (recalling again (5.8)).

We have

$$\iint_{U_\alpha \times U_\beta} [1_{\bar{x} + \zeta^{(1)}_{\bar{x}\bar{y}}(B)}(x) 1_{\bar{y} + \zeta^{(2)}_{\bar{x}\bar{y}}(B)}(y)] d\bar{x} d\bar{y}$$
$$= \iint_{\Omega_{x,y}(B \times B)} d\bar{x} d\bar{y}$$
$$= \iint_{B \times B} \left| \frac{\partial(\Omega^{(1)}_{xy}, \Omega^{(2)}_{xy})}{\partial(x'', y'')} \right| dx'' dy''. \qquad (5.20)$$

It follows from (5.18) and the preceding that

$$\frac{\partial \bar{x}}{\partial x''} = -\frac{\partial \zeta^{(1)}_{\bar{x}\bar{y}}}{\partial x''} + O\Big(R^{-\frac{1}{5}}\Big(\Big|\frac{\partial \bar{x}}{\partial x''}\Big| + \Big|\frac{\partial \bar{y}}{\partial \bar{x}''}\Big|\Big)\Big)$$

and hence

$$\frac{\partial \bar{x}}{\partial x''} = -1 + O(\delta) + O(R^{-\frac{1}{5}}). \tag{5.21}$$

Similarly

$$\frac{\partial \bar{x}}{\partial y''} = O(R^{-\frac{1}{5}}) \tag{5.22}$$

$$\frac{\partial \bar{y}}{\partial x''} = O(R^{-\frac{1}{5}}) \tag{5.23}$$

$$\frac{\partial \bar{y}}{\partial y''} = -1 + O(\delta) + O(R^{-\frac{1}{5}}). \tag{5.24}$$

From (5.21)–(5.24)

$$D\Omega_{x,y} = -1 + O(\delta)$$

implying

$$\frac{\partial(\Omega^{(i)}_{xy}, \Omega^{(2)}_{xy})}{\partial(x'', y'')} = 1 + O(\delta)$$

and

$$(5.20) = \omega(x,y)|B|^2 \tag{5.25}$$

where

$$\omega(x,y) = 1 + O(\delta) \tag{5.26}$$

is a smooth function of $x, y$.

Substituting (5.20), (5.25) in (5.17) gives

$$\Big|\sum_{x\in X, y\in Y} e^{iR\psi(p(a_\alpha+x)-p(a_\beta+y))} \, \omega(x,y)\Big| \lesssim R^{-2\tau}|X|\cdot|Y| + R^{2-\frac{1}{2}\kappa_1+4\tau} \tag{5.27}$$

recalling (5.11).

It remains to remove the function $\omega(x,y)$ in (5.27).

First observe that (5.27) formally implies that

$$\Big|\sum_{x\in X, y\in Y} e^{iR\psi(p(a_\alpha+x)-p(a_\beta+y))}\omega(x,y)u(x)v(y)\Big| < R^{-2\tau}|X|\cdot|Y| + R^{2-\frac{1}{2}\kappa_1+4\tau}$$

$$\tag{5.28}$$

whenever $u, v$ are functions on $\mathbb{R}^2$ satisfying $|u| \leq 1, |v| \leq 1$.

Since $\omega$ is a smooth function of $(x, y)$ satisfying (5.26), it follows that $\frac{1}{\omega} \in L^\infty \hat{\otimes} L^\infty$, thus $\frac{1}{\omega} = \Sigma \lambda_\ell (u_\ell \otimes v_\ell)$ where $\|u_\ell\|_\infty, \|v_\ell\|_\infty \lesssim 1$ and $\Sigma |\lambda_\ell| < C$. Hence, by convexity, (5.28) implies

$$\left| \sum_{x \in X, y \in Y} e^{iR\psi(p(a_\alpha + x) - p(a_\beta + y))} \right| \lesssim R^{-2\tau} |X| \cdot |Y| + R^{2 - \frac{1}{3}\kappa_1} \tag{5.29}$$

taking $\tau > 0$ small enough.

This gives an inequality of the type (5.2). The sets $X \subset U_\alpha, Y \subset U_\beta$ are arbitrary sets of $\frac{1}{\sqrt{R}}$-separated points.

Returning to (4.16), we proved that if $X, Y$ are $\frac{1}{\sqrt{R}}$-separated points in $O$, then

$$\left| \sum_{\substack{x \in XUQ_\alpha \\ y \in Y \cap Q_\beta}} e^{iR\psi(p(x) - p(y))} \right| < R^{-2\tau} |X \cap Q_\alpha| \, |Y \cap Q_\beta| + R^{2 - \frac{1}{3}\kappa_1} \tag{5.30}$$

provided $(\alpha, \beta) \notin \mathcal{W} = \mathcal{W}_{\delta, \delta_1}$ and $p(Q_\alpha) - p(Q_\beta) \subset Z$. Here $\tau, \kappa_1 > 0$ are constants and $\delta = \delta_1^2 = R^{-\varepsilon}, \varepsilon > 0$ sufficiently small.

Summation of (5.30) over $\alpha, \beta$ gives

$$\sum_{\substack{p(Q_\alpha) - p(Q_\beta) \subset Z \\ (\alpha, \beta) \notin \mathcal{W}}} \left| \sum_{\substack{x \in X \cap Q_\alpha \\ y \in Y \cap Q_\beta}} e^{iR\psi(p(x) - p(y))} \right| \lesssim R^{2 - 2\tau} + \delta^{-4} R^{2 - \frac{1}{3}\kappa_1} < R^{2 - \kappa_2}. \tag{5.31}$$

Recalling (4.16), it follows that

$$\left| \sum_{\substack{x \in X, y \in Y \\ p(x) - p(y) \in Z}} e^{iR\psi(p(x) - p(y))} \right| < R^{2 - \kappa_2} + \delta^{-4 + \frac{1}{2}c_1} \max |X \cap B_\delta| \max |Y \cap B_\delta|$$

$$< R^{2 - \kappa_2} + \delta^{\frac{1}{2}c_1} R^2$$

$$< R^{2 - \kappa_3} \tag{5.32}$$

since the points in $X, Y$ are $\frac{1}{\sqrt{R}}$-separated.

Equivalently, considering sets $\mathcal{X}, \mathcal{Y} \subset RS^2$ consisting of $\sqrt{R}$-separated points and such that $\mathcal{X} \cup \mathcal{Y}$ is contained in a cap of size $cR$ ($c > 0$ a constant depending on $\Sigma$) we have

$$\left| \sum_{x \in \mathcal{X}, y \in \mathcal{Y}, x - y \in Z} e^{i\psi(x - y)} \right| < R^{2 - \kappa_3}. \tag{5.33}$$

Thus (conditional to Lemma 5.1) we proved the following

LEMMA 5.2. *Let $\mathcal{X}, \mathcal{Y} \subset RS^2$ consist of $\sqrt{R}$-separated points. Then*

$$\left| \sum_{\substack{x \in \mathcal{X}, y \in \mathcal{Y} \\ x-y \in Z, |x-y| < cR}} e^{i\psi(x-y)} \right| < R^{2-\gamma} \tag{5.34}$$

*for some $c > 0, \gamma > 0$ depending on $\psi$ (hence on $\Sigma$).*

# 6 An Exponential Sum Estimate

We prove the key inequality Lemma 5.1.

Let $R$ be large enough, $0 < |q| < O(1)$ and $S, T \subset [0, R^{-\frac{1}{5}}]$ arbitrary discrete sets of $\frac{1}{\sqrt{R}}$-separated points. Denoting

$$\mathfrak{S} = \sum_{s \in S, t \in T} e^{iR(st+qs^2t^2)} \tag{6.1}$$

application of the Cauchy–Schwartz inequality twice gives

$$|\mathfrak{S}|^4 \leq |S|^2 |T|^2 \left| \sum_{\substack{s,s_1 \in S \\ t,t_1 \in T}} e^{iR((s-s_1)(t-t_1)+q(s^2-s_1^2)(t^2-t_1^2))} \right|$$

$$= |S|^2 |T|^2 \left| \sum_{z,w} e^{i[R^{3/5}z_1 w_1 + qR^{1/5}z_2 w_2]} \mu(z)\nu(w) \right| \tag{6.2}$$

where $z = (z_1, z_2), w = (w_1, w_2)$ and $\mu, \nu$ are discrete measures on $[-1, 1]^2$ defined by

$$\mu(z) = \# \left\{ (s, s_1) \in S \times S \; \middle| \; \begin{array}{l} s - s_1 = R^{-\frac{1}{5}}z_1 \\ s^2 - s_1^2 = R^{-\frac{2}{5}}z_2 \end{array} \right\} \tag{6.3}$$

and similarly for $\nu$. Thus

$$\sum \mu(z), \sum \nu(w) \leq R^{3/5}. \tag{6.4}$$

Fix $0 < \theta < \frac{1}{10}$. Since $S$ is $\frac{1}{\sqrt{R}}$-separated

$$\sum_{|z_1| < R^{-\theta}} \mu(z) < |S| R^{\frac{3}{10} - \theta} \lesssim R^{3/5 - \theta} \tag{6.5}$$

and similarly for $\nu$. Hence in (6.2),

$$
\left| \sum_{z,w} e^{i[R^{3/5}z_1w_1 + qR^{1/5}z_2w_2]} \mu(z)\nu(w) \right|
$$

$$
< \left| \sum_{\substack{z,w \\ |z_1|,|w_1| > R^{-\theta}}} e^{i[R^{3/5}z_1w_1 + qR^{1/5}z_2w_2]} \mu(z)\nu(w) \right| + O(R^{\frac{6}{5}-\theta}). \qquad (6.6)
$$

In order to bound the RHS of (6.6), we apply the following general bilinear estimate.

LEMMA 6.1.

$$
\left| \sum_{z,w} e^{i(R_1 z_1 w_1 + R_2 z_2 w_2)} \mu(z)\nu(w) \right| \lesssim (R_1 R_2)^{\frac{1}{2}} \left( \sum \mu(z) \right)^{\frac{1}{2}} \left( \sum \nu(w) \right)^{\frac{1}{2}}
$$

$$
\cdot \left[ \max_{\xi_1,\xi_2} \mu \left( B\left(\xi_1, \frac{1}{R_1}\right) \times B\left(\xi_2, \frac{1}{R_2}\right) \right) \right]^{\frac{1}{2}}
$$

$$
\cdot \left[ \max_{\xi_1,\xi_2} \nu \left( B\left(\xi_1, \frac{1}{R_1}\right) \times B\left(\xi_2, \frac{1}{R_2}\right) \right) \right]^{\frac{1}{2}}.
$$

*Proof.* Denoting $P_\varepsilon$ an approximate identity on $\mathbb{R}$, the left side equals

$$
\left| \sum_w \hat{\mu}(R_1 w_1, R_2 w_2)\nu(w) \right| \lesssim \sum_w \left| \left( \mu * (P_{\frac{1}{R_1}} \otimes P_{\frac{1}{R_2}}) \right)^{\wedge}(R_1 w_1, R_2 w_2) \right| \nu(w)
$$

$$
\lesssim R_1 R_2 \iint \left| \left( \mu * (P_{\frac{1}{R_1}} \otimes P_{\frac{1}{R_2}}) \right)^{\wedge}(R_1\xi_1, R_2\xi_2) \right| \nu \left( B\left(\xi_1, \frac{1}{R_1}\right) \times B\left(\xi_2, \frac{1}{R_2}\right) \right) d\xi_1 d\xi_2
$$

$$
\leq (R_1 R_2)^{\frac{1}{2}} \|\mu * (P_{\frac{1}{R_1}} \otimes P_{\frac{1}{R_2}})\|_2 \cdot \left\{ \iint \left[ \nu \left( B\left(\xi_1, \frac{1}{R_1}\right) \times B\left(\xi_2, \frac{1}{R_2}\right) \right) \right]^2 d\xi_1 d\xi_2 \right\}^{1/2}
$$

$$
\lesssim (R_1 R_2)^{-\frac{1}{2}} \|\mu * (P_{\frac{1}{R_1}} \otimes P_{\frac{1}{R_2}})\|_2 \|\nu * (P_{\frac{1}{R_1}} \otimes P_{\frac{1}{R_2}})\|_2 \qquad (6.7)
$$

where $\| \ \|_2$ refers to $L^2([-1,1]^2)$.

Next

$$
\|\mu * (P_{\frac{1}{R_1}} \times P_{\frac{1}{R_2}})\|_2 \leq \|\mu * \left( P_{\frac{1}{R_1}} \times P_{\frac{1}{R_2}} \right)\|_1^{\frac{1}{2}} \|\mu * \left( P_{\frac{1}{R_1}} \times P_{\frac{1}{R_2}} \right)\|_\infty^{\frac{1}{2}}
$$

$$
\sim \left[ \sum \mu(z) \right]^{\frac{1}{2}} (R_1 R_2)^{\frac{1}{2}} \left[ \max_\xi \mu \left( B\left(\xi_1, \frac{1}{R_1}\right) \times B\left(\xi_2, \frac{1}{R_2}\right) \right) \right]^{\frac{1}{2}}
$$

$$
\qquad (6.8)
$$

and similarly for $\nu$.

Substitution of (6.8) in (6.7) proves the Lemma.                         $\square$

Now apply Lemma 6.1 to evaluate (6.6). Thus $R_1 = R^{3/5}$, $R_2 = qR^{1/5}$. It remains to bound for $\xi = (\xi_1, \xi_2) \in [-1,1]^2$, $|\xi_1| > R^{-\theta}$, the quantity

$$\mu \left( B \left( \xi_1, \frac{1}{R_1} \right) \times B \left( \xi_2, \frac{1}{R_2} \right) \right)$$
$$= \# \left\{ (s, s_1) \in S \times S; |R^{-\frac{1}{5}}\xi_1 - (s - s_1)| < R^{-4/5} \text{ and } |R^{-\frac{2}{5}}\xi_2 - (s^2 - s_1^2)| < \frac{1}{q}R^{-3/5} \right\}. \tag{6.9}$$

From the equations, one gets

$$\left| R^{-\frac{1}{5}}\frac{\xi_2}{\xi_1} - (s + s_1) \right| < \frac{1}{q}R^{-\frac{2}{5}+\theta} + R^{-\frac{3}{5}+\theta} < \frac{2}{q}R^{-\frac{2}{5}+\theta}$$

and

$$\left| R^{-\frac{1}{5}}\left( \xi_1 + \frac{\xi_2}{\xi_1} \right) - 2s \right| < \frac{3}{q}R^{-\frac{2}{5}+\theta}. \tag{6.10}$$

Since the elements of $S$ are $\frac{1}{\sqrt{R}}$-separated, (6.10) restricts $s$ to at most $\frac{c}{q}R^{\frac{1}{10}+\theta}$ values. Hence

$$|(6.9)| \lesssim \frac{1}{q}R^{\frac{1}{10}+\theta}. \tag{6.11}$$

From Lemma 6.1, recalling (6.4) and (6.11), we obtain

$$|(6.6)| \lesssim (qR^{4/5})^{\frac{1}{2}}R^{3/5}\frac{1}{q}R^{\frac{1}{10}+\theta} \lesssim \frac{1}{\sqrt{q}}R^{\frac{11}{10}+\theta}. \tag{6.12}$$

Hence

$$|(6.2)| \lesssim \frac{1}{\sqrt{q}}R^{\frac{11}{10}+\theta} + R^{\frac{6}{5}-\theta}$$

and

$$|\mathfrak{S}|^4 \lesssim R^{6/5}\left( \frac{1}{\sqrt{q}}R^{\frac{11}{10}+\theta} + R^{\frac{6}{5}-\theta} \right).$$

An appropriate choice of $\theta$ gives

$$|\mathfrak{S}| < R^{\frac{47}{80}}q^{-\frac{1}{4}}. \tag{6.13}$$

This proves Lemma 5.1 with $\kappa_1 = \frac{1}{80}$. □

## 7  Mean Equidistribution Property of Lattice Points

Let $\mathcal{E} = \mathbb{Z}^3 \cap R S^2, R = \sqrt{E}$. Recall that

$$|\mathcal{E}| \ll R^{1+\varepsilon} \quad \text{for all } \varepsilon > 0. \tag{7.1}$$

In order to apply Lemma 5.2 with $\mathcal{X}, \mathcal{Y} \subset \mathcal{E}$, we recall Lemma 1.5, which states that

LEMMA 7.1. Let $\{C_\alpha\}$ be a partition of $R S^2$ in cells of size $\sqrt{R}$. Then

$$\sum_\alpha |C_\alpha \cap \mathcal{E}|^2 \ll R^{1+\varepsilon} \quad \text{for all } \varepsilon > 0. \tag{7.2}$$

Thus (7.2) express the desired separation property in some averaged sense. To obtain sets that are $\sqrt{R}$-separated, proceed as follows. Fix $\varepsilon' > 0$ and let

$$\mathcal{E}' = \bigcup_{|C_\alpha \cap \mathcal{E}| > R^{\varepsilon'}} (\mathcal{E} \cap C_\alpha).$$

It follows from (7.2) that

$$|\mathcal{E}'| < R^{-\varepsilon'} \sum |\mathcal{E} \cap C_\alpha|^2 \ll R^{1+\varepsilon-\varepsilon'} < R^{1-\frac{\varepsilon'}{2}}. \tag{7.3}$$

Also

$$\mathcal{E} \backslash \mathcal{E}' = \bigcup_{s < R^{\varepsilon'}} \mathcal{X}_s \tag{7.4}$$

with each set $\mathcal{X}$ consisting of $\sqrt{R}$-separated points.

From (5.34)

$$\left| \sum_{\substack{x \in \mathcal{X}_s, y \in \mathcal{X}_t \\ x-y \in Z, |x-y| \ll cR}} e^{i\psi(x-y)} \right| < R^{2-\gamma} \tag{7.5}$$

for some $\gamma > 0$.

Therefore, from (7.1), (7.3)

$$\left| \sum_{\substack{x \in \mathcal{E}_1, y \in \mathcal{E}_2 \\ x-y \in Z, |x-y| < cR}} e^{i\psi(x-y)} \right| \ll R^{2-\gamma+2\varepsilon'} + 2|\mathcal{E}'| \, |\mathcal{E}| < R^{2-\gamma+2\varepsilon'} + R^{2+\varepsilon-\frac{\varepsilon'}{2}} \tag{7.6}$$

if $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{E}$.

Hence

LEMMA 7.2. *There is a constant $\gamma_1 > 0$ (independent of $R$) such that*

$$\left| \sum_{\substack{x \in \mathcal{E}_1, y \in \mathcal{E}_2 \\ x-y \in Z, |x-y| < cR}} e^{i\psi(x-y)} \right| < R^{2-\gamma_1} \tag{7.7}$$

*whenever $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{E} = (RS^2 \cap \mathbb{Z}^3)$.*

This is our main estimate to handle 'large distances' $|x - y| > R^{1-\varepsilon}$.

## 8 Restriction Upper Bound

**Theorem 8.1.** *Let $\Sigma \subset \mathbb{T}^3$ be a real-analytic 2-dim submanifold with non-vanishing curvature and let $\sigma$ be its surface measure. There is a constant $K_\Sigma > 0$ such that*

$$\int_\Sigma |\varphi|^2 d\sigma \leq K_\Sigma \|\varphi\|_2^2 \tag{8.1}$$

*for all eigenfunctions $\varphi$ on $\mathbb{T}^3$.*

Let

$$\varphi = \sum_{n \in \mathcal{E}} a_n e^{ix.n} \quad \text{with} \quad \sum |a_n|^2 = 1 \tag{8.2}$$

and

$$\mathcal{E} = \{n \in \mathbb{Z}^3; |n|^2 = E = R^2\}.$$

Then

$$\int_\Sigma |\varphi|^2 d\sigma = \sum_{m,n \in \mathcal{E}} a_m \bar{a}_n \int_\Sigma e^{i(m-n).x} \sigma(dx)$$

$$= \|\varphi\|_2^2 \text{area}(\Sigma) + \sum_{k \geq 0} \sum_{\substack{m,n \in \mathcal{E} \\ 2^k \leq |m-n| < 2^{k+1}}} a_m \bar{a}_n \int_\Sigma e^{i(m-n).x} \sigma(dx). \tag{8.3}$$

Considering local coordinate charts, we can assume $\Sigma$ is parametrized as in (3.1). From (3.10), if $m \neq n$ then

$$\int_\Sigma e^{i(m-n).x} d\sigma = \frac{1}{|m-n|} \eta \left( \frac{m-n}{|m-n|} \right) e^{i\psi(m-n)} + O\left( \frac{1}{|m-n|^2} \right) \tag{8.4}$$

with $\eta$ a smooth function on $S^2$.

First we bound the contribution of the error term in (8.4) by writing

$$\sum_{m,n\in\mathcal{E},2^k\leq|m-n|<2^{k+1}} |a_m|\,|a_n|\frac{1}{|m-n|^2} \lesssim 4^{-k}\sum_\alpha \left(\sum_{m\in\mathcal{E}\cap C_\alpha}|a_m|\right)^2$$

$$\leq 4^{-k}\big(\max_\alpha|\mathcal{E}\cap C_\alpha|\big)\sum_\alpha\left(\sum_{m\in\mathcal{E}\cap C_\alpha}|a_m|^2\right)$$

$$\lesssim 4^{-k}\max_\alpha|\mathcal{E}\cap C_\alpha| \tag{8.5}$$

where $\{C_\alpha\}$ is a partition of $RS^2$ in cells of size $\sim 2^k$. Thus we need to bound $|\mathcal{E}\cap C_r|$, where $C_r\subset RS^2$ is a cap of size $r$.

If $r<cR^{1/4}$, a Jarnik type theorem implies that $\mathcal{E}\cap C_r$ lies in a 2-dim affine plane $H$. Projection of $H\cap RS^2$ on one of the coordinate planes $xy,yz,zx$ gives a non-degenerate ellipse of size $\sim r$. Another application of the classical Jarnik theorem in the plane shows that certainly

$$|\mathcal{E}\cap C_r| < Cr^{2/3}. \tag{8.6}$$

For $r$ arbitrary, one has the (easy) linear bound (see Lemma 2.2)

$$|\mathcal{E}\cap C_r| \ll R^\varepsilon(1+r). \tag{8.7}$$

From (8.6), (8.7) we get

$$|\mathcal{E}\cap C_r| \ll r^{1+\varepsilon}. \tag{8.8}$$

Substituting (8.8) in (8.5) gives $2^{-k(1-\varepsilon)}$, which is summable in $k$.

Consider next the contribution of the main term in (8.4). We make two separate estimates. The first treats the case $2^k < R^{1-\varepsilon_0}$ ($\varepsilon_0>0$ some small constant) and the second $2^k > R^{1-\varepsilon_0}$. The following improvement of the lattice point estimates (8.6), (8.7), which will be proven in Sect. 2 (Lemma 2.3), is crucial to our analysis: For $0<\eta<1/16$

$$|\mathcal{E}\cap C_r| \ll R^\varepsilon\Big(1+r\Big(\frac{r}{R}\Big)^\eta\Big). \tag{8.9}$$

**The case** $2^k < R^{1-\varepsilon_0}$ Ignoring again the phase function, and arguing as in (8.5) gives

$$\sum_{m,n\in\mathcal{E},2^k\leq|m-n|<2^{k+1}} |a_m|\,|a_n|\frac{1}{|m-n|} \lesssim 2^{-k}\max_\alpha|\mathcal{E}\cap C_\alpha|$$

$$< \begin{cases} C2^{-\frac{1}{3}k} & \text{if } 2^k < cR^{1/4} \\ R^\varepsilon\Big[2^{-k}+\Big(\frac{2^k}{R}\Big)^\eta\Big] & \text{if } cR^{1/4} < 2^k < R^{1-\varepsilon_0} \end{cases} \tag{8.10}$$

invoking (8.6), (8.9). These bounds are again conclusive.

**The case** $R^{1-\varepsilon_0} < 2^k < R$ This requires a more subtle argument involving the oscillatory factor $e^{i\psi(n-n)}$ in (8.4).

Let $b$ be a smooth (radial) function on $\mathbb{R}^3$ satisfying

$$\begin{cases} b(x) = 0 & \text{if } |x| < \frac{1}{2} \\ b(x) = 1 & \text{if } |x| > 1 \end{cases}$$

and estimate

$$\sum_{m,n\in\mathcal{E}} a_m \bar{a}_n \, b\left(\frac{m-n}{R^{1-\varepsilon_0}}\right) \frac{1}{|m-n|} \eta\left(\frac{m-n}{|m-n|}\right) e^{i\psi(m-n)}. \tag{8.11}$$

Decompose $\mathcal{E} = \mathcal{E}_\ell \coprod \mathcal{E}_s$, where

$$\mathcal{E}_\ell = \{m \in \mathcal{E}; |a_m| > R^{-\frac{1}{2}+2\varepsilon_0}\}.$$

Since $\sum_n |a_n|^2 = 1$, we have $|\mathcal{E}_\ell| \leq R^{1-4\varepsilon_0}$.

Then we estimate

$$|(8.11)| \lesssim \sum_{(m,n)\in(\mathcal{E}\times\mathcal{E})\setminus(\mathcal{E}_s\times\mathcal{E}_s)} |a_m|\,|a_n|\frac{1}{R^{1-\varepsilon_0}} \tag{8.12}$$

$$+ \left| \sum_{m,n\in\mathcal{E}_s} a_m \bar{a}_n \, b\left(\frac{m-n}{R^{1-\varepsilon_0}}\right) \frac{1}{|m-n|} \eta\left(\frac{m-n}{|m-n|}\right) e^{i\psi(m-n)} \right|. \tag{8.13}$$

By Cauchy–Schwarz, (8.12) is bounded by

$$\frac{1}{R^{1-\varepsilon_0}}|\mathcal{E}|^{\frac{1}{2}}\,|\mathcal{E}_\ell|^{\frac{1}{2}} \ll \frac{1}{R^{1-\varepsilon_0}}R^{\frac{1}{2}+o(1)}R^{\frac{1}{2}-2\varepsilon_0} < R^{-\frac{\varepsilon_0}{2}}.$$

The term (8.13) is bounded using Lemma 7.2 and a partition of unity. This gives an estimate of the form

$$R^{C\varepsilon_0}\frac{1}{R^{1-\varepsilon_0}}\frac{1}{R^{1-4\varepsilon_0}}R^{2-\gamma_1} < R^{-\frac{1}{2}\gamma_1}$$

if $\varepsilon_0$ is chosen sufficiently small.

Hence, we have proven Theorem 8.1. $\qquad\qquad\square$

# 9  Restriction Lower Bounds

We prove

**Theorem 9.1.**  *Given $\Sigma$ as in Theorem 8.1, there is $E_0 \in \mathbb{Z}_+$ and a constant $k_\Sigma > 0$ such that*

$$\int_\Sigma |\varphi|^2 d\sigma \geq k_\Sigma ||\varphi||_2^2 \tag{9.1}$$

*whenever $\varphi$ is an eigenfunction with eigenvalue $E > E_0$.*

Let $\varphi = \sum_{n \in \mathcal{E}} a_n e^{in.x}$ with $\mathcal{E} = RS^2 \cap \mathbb{Z}^3, R = \sqrt{E}$ and $\Sigma |a_n|^2 = 1$.
Write

$$\int_\Sigma |\varphi|^2 d\sigma = \sum_{\substack{m,n \in \mathcal{E} \\ |m-n| < R^{\frac{1}{5}}}} a_m \bar{a}_n \int_\Sigma e^{i(m-n)x} \sigma(dx) \tag{9.2}$$

$$+ \sum_{|m-n| > R^{\frac{1}{5}}} \cdots \tag{9.3}$$

$$= (9.2) + (9.3).$$

From the upper-bound analysis in Sect. 8, we have

$$|(9.3)| < R^{-\delta} \tag{9.4}$$

for some $\delta > 0$.

Next, we analyze (9.2).

Introduce a graph $\mathcal{G}$ on $\mathcal{E}$ defined by

$$\mathcal{G} = \{(m,n) \in \mathcal{E}, |m-n| < R^{1/5}\}.$$

Let $\{\mathcal{E}_\alpha\}$ be the connected components of $\mathcal{G}$.

LEMMA 9.2. *For each $\alpha$, the set $\mathcal{E}_\alpha$ is contained in an affine plane.*

*Proof.* We may obviously assume $\#\mathcal{E}_\alpha \geq 3$ and hence there is a subset $\mathcal{F}_0 \subset \mathcal{E}_\alpha, \#\mathcal{F}_0 = 3$ and $\operatorname{diam} \mathcal{F}_0 < 2R^{1/5}$.

Let $H = \langle \mathcal{F}_0 \rangle$ be the affine plane spanned by $\mathcal{F}_0$.

Write

$$\mathcal{E}_\alpha = \bigcup_j \mathcal{F}_j$$

where

$$\mathcal{F}_{j+1} = \{m \in \mathcal{E}; \operatorname{dist}(m, \mathcal{F}_j) < R^{1/5}\}.$$

We show inductively that $\mathcal{F}_j \subset H$ for each $j$.

For $j < R^{\frac{1}{100}}$, $\operatorname{dist}(\mathcal{F}_j, \mathcal{F}_0) < jR^{1/5} \ll R^{1/4}$ and Jarnik's theorem implies that $\mathcal{F}_j$ is coplanar. Hence $\mathcal{F}_j \subset H$. Next, assume $j_0 \geq R^{\frac{1}{100}}, \mathcal{F}_{j_0} \subset H$ and $\mathcal{F}_{j_0+1} \neq \mathcal{F}_{j_0}$. If $x_{j_0+1} \in \mathcal{F}_{j_0+1}$, there are clearly $x \in \mathcal{F}_{j_0}$ and $y, z \in \mathcal{F}_{j_0}$ satisfying

$$|x_{j_0+1} - x| < R^{\frac{1}{5}}$$

and

$$x, y, z \text{ are distinct and } \operatorname{diam}\{x, y, z\} \lesssim R^{1/5}.$$

(we use here that $\#\mathcal{F}_0 = 3$).

Since diam $\{x, y, z, x_{j_0+1}\} \lesssim R^{\frac{1}{5}}$, it follows again from Jarnik that $x, y, z, x_{j_0+1}$ are coplanar and hence

$$x_{j_0+1} \in \langle x, y, z \rangle = H.$$

This proves Lemma 9.2. $\hfill\square$

Returning to (9.2), it follows from definition of $\mathcal{G}$ that

$$(9.2) = \sum_{\alpha} \sum_{\substack{m,n \in \mathcal{E}_\alpha \\ |m-n| < R^{1/5}}} a_m \bar{a}_n \int_{\Sigma} e^{i(m-n)x} d\sigma$$

$$= \sum_{\alpha} \int_{\Sigma} \Big| \sum_{m \in \mathcal{E}_\alpha} a_m\, e^{imx} \Big|^2 d\sigma$$

$$+ 0 \left( \sum_{\alpha} \sum_{\substack{m,n \in \mathcal{E}_\alpha \\ |m-n| \geq R^{1/5}}} \frac{|a_m|\,|a_n|}{|m-n|} \right). \qquad (9.5)$$

The last term may be bounded by $R^{-\frac{1}{31}}$, as seen as follows. Estimate by

$$(9.5) < \sum_{2^k > R^{1/5}} 2^{-k} \Big( \max_C |\{(m,n) \in C \times C; m, n \in \mathcal{E}_\alpha \text{ for some } \alpha\}| \Big)^{\frac{1}{2}}$$

where the max is taken over all $2^k$-caps $C$. For $C$ an $r$-cap, (8.6), (8.8) imply that

$$|\{\cdots\}| \leq |C \cap \mathcal{E}| \cdot \max_{\alpha} |C \cap \mathcal{E}_\alpha| \ll r^{1+2/3+\varepsilon}$$

hence the claim.

To prove Theorem 9.1, it will therefore suffice to show the following:

LEMMA 9.3. Let $\varphi = \sum_{m \in \mathcal{F}} a_m e^{im.x}$, $\sum_{m \in \mathcal{F}} |a_m|^2 = 1$, where $\mathcal{F} \subset \mathcal{E}$ consists of coplanar points. Then

$$\int_{\Sigma} |\varphi|^2 d\sigma > k \qquad (9.6)$$

where $k > 0$ is independent of $E$.

*Proof.* Let $H = \langle \mathcal{F} \rangle$ be the plane containing $\mathcal{F}$ and $\pi$ the orthogonal projection on $H_0$=plane parallel to $H$ through 0. Clearly, fixing any element $m_0 \in \mathcal{F}$, we have

$$\int_{\Sigma} \Big| \sum_{m \in \mathcal{F}} a_m e^{imx} \Big|^2 d\sigma = \int_{\Sigma} \Big| \sum_{m \in \mathcal{F}} a_m e^{i(m-m_0).x} \Big|^2 d\sigma$$

$$= \int_{\Sigma} \Big| \sum_{m \in \mathcal{F}} a_m e^{i(m-m_0).\pi(x)} \Big|^2 d\sigma. \qquad (9.7)$$

Let $\pi[\sigma]$ be the image measure of $\sigma$ under the map $\pi\big|_\Sigma : \Sigma \to H_0$. Since $\Sigma$ has non-vanishing curvature, there is a disc $B_\rho \subset H_0$ ($\rho$-independent of $H_0$) such that

$$\pi[\sigma] \geq \mu_{H_0}\big|_{B_\rho} \tag{9.8}$$

where $\mu_{H_0}$ is a Lebesque measure on $H_0$. Hence

$$(9.7) \geq \int_{B_\rho} \left| \sum_{m \in \mathcal{F}} a_m e^{i(m-m_0).y} \right|^2 dy. \tag{9.9}$$

Since $\mathcal{F} \subset RS^2 \cap H, \mathcal{F}_0 = \mathcal{F} - m_0$ lies on a translate of some circle $\{x \in H_0; |x| = r\}, r \leq R$.

Let $r_0$ be sufficiently large (to be specified later).

We distinguish two cases.

**Case 1:** $r < r_0$

Since $\Sigma_{m \in \mathcal{F}} a_m e^{i(m-m_0)\cdot y}$ is a nonzero trigonometric polynomial with frequencies $|m - m_0| < r_0$, it follows that

$$(9.9) > C(\rho, r_0). \tag{9.10}$$

**Case 2:** $r \geq r_0$.

By Jarnik's theorem,

$$\mathcal{F}_0 = \bigcup_\alpha \mathcal{F}_\alpha$$

where $\#\mathcal{F}_\alpha \leq 2$ and $\mathrm{dist}\,(\mathcal{F}_\alpha, \mathcal{F}_\beta) \gtrsim r^{1/3}$ for $\alpha \neq \beta$. Let $\eta$ be a smooth bumpfunction, supp $\eta \subset B_\rho$. Then

$$(9.9) \geq \int \left| \sum_{m \in \mathcal{F}} a_m e^{i(m-m_0).y} \right|^2 \eta(y) dy$$

$$= \sum_\alpha \int \left| \sum_{m \in \mathcal{F}_\alpha} a_m e^{i(m-m_0).y} \right|^2 \eta(y) dy \tag{9.11}$$

$$+ \sum_{\substack{\alpha \neq \beta \\ m \in \mathcal{F}_\alpha, n \in \mathcal{F}_\beta}} a_m \bar{a}_n \int e^{i(m-n).y} \eta(y) dy \tag{9.12}$$

and

$$|(9.12)| \leq \sum_{\substack{m,n \in \mathcal{F} \\ |m-n| \gtrsim r^{1/3}}} |a_m| \, |a_n| \frac{C(\rho)}{|m-n|^{10}} < \frac{C(\rho)}{r_0}. \tag{9.13}$$

Since $\#\mathcal{F}_\alpha \leq 2$, arguing as in the proof of the case $d = 2$ (see the Introduction), we have for each $\alpha$

$$\int \left| \sum_{m \in \mathcal{F}_\alpha} a_m e^{i(m-m_0).y} \right|^2 \eta(y) dy > c(\rho) \sum_{m \in \mathcal{F}_\alpha} |a_m|^2 \tag{9.14}$$

and thus

$$(9.11) > c(\rho) \left( \sum_{m \in \mathcal{F}} |a_m|^2 \right) = c(\rho). \tag{9.15}$$

From (9.11)–(9.15)

$$(9.9) > c(\rho) - \frac{C(\rho)}{r_0} > \frac{1}{2} c(\rho)$$

for appropriate $r_0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

This concludes the proof of Theorem 9.1.

## 10   Intersection of Nodal Sets with Submanifolds

We start by recalling the following result from [BR11a].

**Theorem 10.1** ([BR11a]).   *Let $\Sigma \in \mathbb{T}^d$ be a real analytic, codimension one, hypersurface with nowhere-vanishing Gauss curvature. Then there is some $E_\Sigma > 0$ so that if $E > E_\Sigma$, then $\Sigma$ cannot be part of the nodal set of any eigenfunction $\varphi_E$ with eigenvalue $E$.*

The reader is referred to [BR11a] for a discussion of this phenomenon. Our aim here is to prove a quantitative version. Denote $h_s(A)$ the $s$-dimensional Hausdorff measure of the set $A$.

**Theorem 10.2.**   *Let $\Sigma$ be as above, $E > E_\Sigma$ and $\varphi_E$ an eigenfunction of $\mathbb{T}^d$ with eigenvalue $E$. Let $N$ denote the nodal set of $\varphi_E$. Then*

$$h_{d-2}(N \cap \Sigma) < C_\Sigma \sqrt{E}. \tag{10.1}$$

Recall at this point also the Donnelly–Fefferman theorem, stating that if $M$ is a real-analytic $d$-dimensional manifold and $\varphi_E$ an eigenfunction
$-\Delta \varphi = E\varphi, \Delta$ the Laplacian of $M$, then the nodal set $N$ of $\varphi_E$ satisfies

$$h_{d-1}(N) < C\sqrt{E} \tag{10.2}$$

where $C = C(M)$. See [DF88].

As in [DF88], we will establish (10.1) combining Jensen's inequality and Crofton's formula. Of course, an additional ingredient is needed, namely some type of lower bound on the restriction $\varphi|_\Sigma$.

First recall some basic facts on one-variable analytic functions.

LEMMA 10.3. *Let $f$ be a bounded analytic function on the unit disc $D = \{|z| < 1\}$. Let $a \in D_{\frac{1}{2}} = \{|z| < 1\}$ such that $f(a) \neq 0$ and denote $\nu(D_{\frac{1}{2}})$ the number of zeros of $f$ on $D_{\frac{1}{2}}$. Then*

$$\nu(D_{\frac{1}{2}}) \leq C\left(\left|\log|f(a)|\right| + \log\left[\sup_{z \in D}|f(z)|\right]\right). \tag{10.3}$$

Hence

LEMMA 10.4. *Let $f \neq 0$ be a real analytic function on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ with bounded analytic extension to $D$. Let $\nu$ be the number of zeros of $f$. Then*

$$\nu \leq C\left(\min_{x \in [-\frac{1}{2}, \frac{1}{2}]}\left|\log|f(x)|\right| + \log\left[\sup_{z \in D}|\tilde{f}(z)|\right]\right). \tag{10.4}$$

LEMMA 10.5. *Let $f$ be as in Lemma 10.4. Then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\log|f(x)|\right|dx \leq C\min_{a \in D_{\frac{1}{2}}}\left|\log|\tilde{f}(a)|\right| + \log\left[\sup_{z \in D}|\tilde{f}(z)| + 1\right]. \tag{10.5}$$

Lemma 10.3 follows from Jensen's theorem and (10.5) is easily deduced from subharmonicity of $\log|\tilde{f}(z)|$.

Lemma 10.5 generalizes to real analytic functions of several variables.

LEMMA 10.6. *Let $f \neq 0$ be a real analytic function on $[-\frac{1}{2}, \frac{1}{2}]^m$, $m \geq 1$ with bounded analytic extension $\tilde{f}$ to the polydisc $D^m$. Denote*

$$M = \sup_{z \in D^m}|\tilde{f}(z)| + 1.$$

*Then*

$$\int_{[-\frac{1}{2}, \frac{1}{2}]^m}\left|\log|f(x)|\right|dx \leq C\left(\min_{a \in D_{\frac{1}{2}}^m}\left|\log|\tilde{f}(a)|\right| + \log M\right). \tag{10.6}$$

*Proof.* Fix $a \in D_{\frac{1}{2}}^m$. From (10.5) applied to the function $\tilde{f}(\cdot, a_2, \ldots, a_m)$, we get

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\log|\tilde{f}(x_1, a_2, \ldots, a_m)|\right|dx_1 \leq C\left|\log|\tilde{f}(a)|\right| + C\log M. \tag{10.7}$$

Next, fix $|x_1| < \frac{1}{2}$ and apply (10.5) to the function $\tilde{f}(x_1, ., a_3, \ldots, a_m)$. Hence

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\log|\tilde{f}(x_1, x_2, a_3, \ldots, a_m)|\right|dx_2 \leq C\left|\log|\tilde{f}(x_1, a_2, \ldots, a_m)|\right| + C\log M. \tag{10.8}$$

Integrating (10.8) in $x_1$ and using (10.7) gives

$$\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \big| \log|\tilde{f}(x_1, x_2, a_3, \ldots, a_m)|\big| dx_1 dx_2 \leq C \big| \log|\tilde{f}(a)|\big| + C \log M. \quad (10.9)$$

Iteration yields (10.6). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

LEMMA 10.7. *Let $f$ be as in Lemma 10.6 and*

$$Z = \Big\{ x \in \Big[ -\frac{1}{2}, \frac{1}{2}\Big]^m ; f(x) = 0 \Big\}$$

*Then*

$$h_{m-1}(Z) \leq C \left( \min_{a \in D_{\frac{1}{2}}^m} \big| \log|\tilde{f}(a)|\big| + \log M \right). \quad (10.10)$$

*Proof.* For $m = 1$, (10.10) follows from (10.3).

For $m > 1$, we use Crofton's formula

$$h_{m-1}(Z) \sim \int_{\mathcal{L}} [\#(Z \cap \ell)] d\ell \quad (10.11)$$

where $\mathcal{L} \simeq G_{m,1} \times \mathbb{R}^m$ is the space of affine straight lines $\ell$.

Fix $\ell \in \mathcal{L}, \ell \cap Z \neq \phi$ and let $\ell = b + \mathbb{R}\xi, b \in [-\frac{1}{2}, \frac{1}{2}]^m, |\xi| = 1$. Denote $I$ the interval $I = \{x \in \mathbb{R}; b + x\xi \in [-\frac{3}{4}, \frac{3}{4}]^m\}$. Let

$$g(x) = f(b + x\xi)$$

which is real analytic with analytic extension $\tilde{g}$ to $\{z \in \mathbb{C}; \text{dist}(z, I) < \frac{1}{2}\}$ bounded by $\log M$. Lemma 10.4 implies

$$\#[Z \cap \ell] \leq \#\{x \in I; g(x) = 0\}$$

$$\leq c \min_{x \in I} \big| \log|g(x)|\big| + C \log M$$

$$\leq c \int_{\ell \cap [-\frac{3}{4}, \frac{3}{4}]^m} \big| \log|f(x)|\big| + C \log M. \quad (10.12)$$

Integration of (10.12) over $\mathcal{L}$ and invoking Lemma 10.6 gives

$$h_{m-1}(Z) \leq c \int_{[-\frac{3}{4}, \frac{3}{4}]^m} \big| \log|f(x)|\big| + C \log M \leq (10.10)$$

proving Lemma 10.7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

*Proof of Theorem 10.2.* Let $p : Q \to \Sigma$ be a real analytic parametrization of $\Sigma$ where $Q$ is an $(d-1)$-dimensional interval. Thus

$$N \cap \Sigma = p\big(\{x \in Q; \varphi\big(p(x)\big) = 0\}\big).$$

Since $\varphi(x) = \varphi_E(x) = \underset{\xi \in \mathbb{Z}^d, |\xi|^2 = E}{\Sigma} \hat{\varphi}(\xi)e(x.\xi)$, the analytic extension

$$\tilde{\varphi}(z_1, \ldots, z_d) = \sum \hat{\varphi}(\xi)e(z_1.\xi_1 + \cdots + z_d\xi_d)$$

of $\varphi$ to the polydisc $D_2^d$ obviously admits a bound

$$|\tilde{\varphi}| < E^{\frac{d}{2}}e^{2\sqrt{d}\sqrt{E}} = M \tag{10.13}$$

(assuming $\|\varphi\|_2 = 1$). Thus $\varphi \circ p$ has an analytic extension to a complex neighborhood of $Q$, bounded by $M$.

From Lemma 10.7

$$
\begin{aligned}
h_{d-2}(N \cap \Sigma) &\sim h_{d-2}[x \in Q; \varphi\big(p(x)\big) = 0] \\
&\leq c \min_{a \in \tilde{Q}} \big| \log |(\varphi \circ p)^{\sim}(a)| \big| + c\sqrt{E}
\end{aligned}
\tag{10.14}
$$

where $\tilde{Q} \subset \mathbb{C}^{d-1}$ is some complex neighborhood of $Q$.

For $d = 2$ or $d = 3$, our restriction theorem (lower bounds), assuming $E > E_\Sigma$, implies

$$\max_{x \in \Sigma} |\varphi(x)| \gtrsim \left( \int_\Sigma |\varphi|^2 d\sigma \right)^{\frac{1}{2}} > c_\Sigma \tag{10.15}$$

therefore

$$\log E \gtrsim \log |\varphi\big(p(a)\big| > -c$$

for some $a \in Q$.

For general dimension $d$, we do not have at this point a lower bound of the type (10.15). However, the proof of the result in [BR11a] cited in the beginning of this section, which uses the complexification $(\varphi \circ p)^{\sim}$, implies in fact that for $E > E_\Sigma$

$$\max_{a \in \tilde{Q}} |(\varphi \circ p)^{\sim}(a)| > E^{-C} \tag{10.16}$$

where $C$ is some constant. Hence (10.14) can be applied to obtain (10.1). This proves Theorem 10.2.

REMARK. Theorem 10.2 should be compared with results in [TZ09] ($d = 2$). Using the [TZ09] terminology, a real analytic hypersurface $\Sigma \subset \mathbb{T}^d$ with nowhere vanishing curvature is 'good' in the since that

$$\frac{\max |\varphi|\Big|_\Sigma}{\|\varphi\|_2} > c_\Sigma^{-\sqrt{E}} \tag{10.17}$$

for $\varphi = \varphi_E, E > E_\Sigma$.

The remainder of this section deals with the converse phenomenon.

We show for $d = 2, 3$ that if $\Sigma \subset \mathbb{T}^d$ is as above and $E > E_\Sigma$, $\varphi = \varphi_E$ an eigenfunction with nodal set $N$, then

$$N \cap \Sigma \neq \phi.$$

For $d = 2$, there is a more precise statement.

**Theorem 10.8.** *Let $\Sigma \subset \mathbb{T}^2$ be a real analytic curve which is not geodesic. Let $E \geq E_\Sigma$ and $\varphi_E$ an eigenfunction with eigenvalue $E$ and nodal set $N$. Then*

$$c_\varepsilon E^{\frac{1}{2}-\varepsilon} < \#(N \cap \Sigma) < CE^{\frac{1}{2}} \quad \text{for all } \varepsilon > 0. \tag{10.18}$$

*Proof.* The upper bound follows from (the proof of) Theorem 10.2, noting that since $\Sigma$ is not a straight line segment, there is $\Sigma' \subset \Sigma$ with non-vanishing curvature. For the lower bound, we can replace $\Sigma$ by $\Sigma'$ and proceed as follows.

Fix $\rho = \frac{1}{2} - \varepsilon_0$ and decompose

$$\Sigma = \bigcup_{\alpha \lesssim E^\rho} \Sigma_\alpha$$

in arcs $\Sigma_\alpha$ of size $E^{-\rho}$. From the lower bound ($\|\varphi\|_2 = 1$)

$$c_\Sigma = \int_\Sigma |\varphi|^2 d\sigma = \sum_\alpha \int_{\Sigma_\alpha} |\varphi|^2 d\sigma \tag{10.19}$$

and the upper bound $\|\varphi\|_\infty \ll E^\varepsilon$ for all $\varepsilon > 0$, one easily sees that

$$\#\mathcal{F} = \#\left\{\alpha; \int_{\Sigma_\alpha} |\varphi|^2 d\sigma > cE^{-\rho}\right\} \gg E^{\rho-\varepsilon} \quad \text{for all } \varepsilon > 0. \tag{10.20}$$

For $\alpha \in \mathcal{F}$

$$\int_{\Sigma_\alpha} |\varphi| d\sigma > c\frac{E^{-\rho}}{\|\varphi\|_\infty}. \tag{10.21}$$

Hence, if

$$\int\limits_{\Sigma_\alpha} \varphi d\sigma = o\left(\frac{E^{-\rho}}{\|\varphi\|_\infty}\right) \tag{10.22}$$

we can conclude that $N \cap \Sigma_\alpha \neq \phi$.

Let $\mathcal{E} = \{\xi \in \mathbb{Z}^2; |\xi|^2 = E\}$ and $\varphi = \Sigma_{\xi \in \mathcal{E}} \hat{\varphi}(\xi) e(x \cdot \xi)$, $\|\varphi\|_2 = 1$.

Fix $\varepsilon_1 > 0$ a small number and define

$$\mathcal{F}_1 = \left\{\alpha \in \mathcal{F}; \min_{\xi \in \mathcal{E}} |\vec{t}.\xi| > E^{\frac{1}{2}-\varepsilon_1} \text{ for all tangent vectors } \vec{t} \text{ of } \Sigma_\alpha\right\}. \tag{10.23}$$

Clearly

$$\#(\mathcal{F}\backslash\mathcal{F}_1) \lesssim (\#\mathcal{E}) \frac{E^{-\varepsilon_1}}{E^{-\rho}} < E^{\rho-\frac{\varepsilon_1}{2}}$$

and

$$\#\mathcal{F}_1 > \frac{1}{2}(\#\mathcal{F}) \gg E^{\rho-\varepsilon} \tag{10.24}$$

by (10.20). Next, letting $\gamma : I = [0, E^{-\rho}] \to \Sigma_\alpha$ be an arclength parametrization of $\Sigma_\alpha, \alpha \in \mathcal{F}_1$, write

$$\left|\int\limits_{\Sigma_\alpha} \varphi d\sigma\right| \leq \sum_{\xi \in \mathcal{E}} |\hat{\varphi}(\xi)| \left|\int\limits_I e\big(\xi \cdot \gamma(t)\big) dt\right|$$

and by partial integration

$$\left|\int\limits_I e\big(\xi \cdot \gamma(t)\big) dt\right| \lesssim \max_{t \in I} \frac{1}{|\xi \cdot \dot{\gamma}(t)|} + \int\limits_I \frac{|\xi \cdot \ddot{\gamma}(t)|}{|\xi \cdot \dot{\gamma}(t)|^2} dt \lesssim \frac{E^{\varepsilon_1}}{\sqrt{E}}$$

from the definition of $\mathcal{F}_1$. Hence, for $\alpha \in \mathcal{F}_1$

$$\left|\int\limits_{\Sigma_\alpha} \varphi d\sigma\right| \ll \frac{E^{\varepsilon_1+\varepsilon}}{\sqrt{E}} \quad \text{for all } \varepsilon > 0 \tag{10.25}$$

and (10.22) will hold if $\varepsilon_0 > \varepsilon_1$ and $E$ large enough.

It follows that for $E > E_{\Sigma,\varepsilon_0}$

$$\#(N \cap \Sigma) \geq (\#\mathcal{F}_1) > E^{\frac{1}{2}-2\varepsilon_0}$$

proving Theorem 10.8. □

For $d = 3$, we can show

**Theorem 10.9.** *Let $\Sigma \subset \mathbb{T}^3$ be a real analytic surface with non-vanishing curvature. There is $E_\Sigma$ such that if $E > E_\Sigma$, $E \neq 0, 4, 7 \mod 8$ and $N$ is the nodal set of $\varphi_E$, then*

$$N \cap \Sigma \neq \phi. \tag{10.26}$$

The argument allows more precise statements that we do not attempt to formulate here.

As before, (10.26) will be derived from a property

$$\left| \int_{\Sigma} \varphi(x)\omega(x)d\sigma \right| = o\left( \int_{\Sigma} |\varphi(x)\,\omega(x)|\,d\sigma \right) \tag{10.27}$$

with $0 \leq \omega \leq 1$ a smooth localizing function on $\Sigma$.

Letting

$$\phi(x) = \sum_{\xi \in \mathcal{E}} \hat{\phi}(\xi)e(x \cdot \xi), \qquad \mathcal{E} = \{\xi \in \mathbb{Z}^3; |\xi|^2 = E\}.$$

Then (3.10) allows to bound the left side of (10.27) by ($\|\phi\|_2 = 1$)

$$\sum_{\xi \in \mathcal{E}} |\hat{\phi}(\xi)| \left| \int_{\Sigma} e(x \cdot \xi)\omega(x)d\sigma \right| \lesssim \sum_{\xi \in \mathcal{E}} \frac{|\hat{\phi}(\xi)|}{\sqrt{E}}$$

$$\lesssim E^{-\frac{1}{2}}(\#\mathcal{E})^{\frac{1}{2}} \ll E^{-\frac{1}{4}+\varepsilon}. \tag{10.28}$$

According to Theorem 9.1,

$$\int_{\Sigma} |\varphi|^2 \omega d\sigma > c \tag{10.29}$$

and hence, certainly

$$\int_{\Sigma} |\varphi|\omega d\sigma > \frac{c}{\|\varphi\|_\infty} > \left( \sum |\hat{\varphi}(\xi)| \right)^{-1} \gg E^{-\frac{1}{4}-\varepsilon}$$

which is barely insufficient to conclude.

Instead of interpolating $L^2(\Sigma, d\sigma)$ between $L^1(\Sigma, d\sigma)$ and $L^\infty(\Sigma, d\sigma)$, interpolate $L^2(\Sigma, d\sigma)$ between $L^1(\Sigma, d\sigma)$ and $L^4(\Sigma, d\sigma)$

$$c < \int_{\Sigma} |\varphi|^2 \omega d\sigma \leq \left( \int |\varphi|\omega d\sigma \right)^{\frac{2}{3}} \left( \int |\varphi|^4 \omega d\sigma \right)^{\frac{1}{3}} \tag{10.30}$$

reducing to problem to establish a bound of the form

$$\int_{\Sigma} |\varphi|^4 \omega d\sigma < E^{\frac{1}{2}-\varepsilon_0} \tag{10.31}$$

for some $\varepsilon_0 > 0$.

Note that from Theorem 8.1

$$\int_{\Sigma} |\varphi|^2 \omega d\sigma < C \tag{10.32}$$

and therefore

$$\int\limits_{\Sigma} |\varphi|^4 \omega d\sigma < C\|\varphi\|_\infty^2 \le C\Big(\sum |\hat\varphi(\xi)|\Big)^2 \ll E^{\frac{1}{2}+\varepsilon}. \tag{10.33}$$

Decomposing $\varphi = \varphi_1 + \varphi_2$, with

$$\varphi_1(x) = \sum_{|\hat\varphi(\xi)| > E^{-\frac{1}{4}+\varepsilon_1}} \hat\varphi(\xi)e(x\cdot\xi)$$

the bound (10.33) implies that

$$\int\limits_{\Sigma} |\varphi_1|^4 \omega d\sigma < E^{\frac{1}{2}-2\varepsilon_1}$$

and hence we may assume

$$|\hat\varphi(\xi)| < E^{-\frac{1}{4}+}. \tag{10.34}$$

Fix $\rho = \frac{1}{2} - \tau$, $\tau > 0$ sufficiently small, and partition

$$\mathcal{E} = \bigcup_\alpha \mathcal{E}_\alpha$$

in $\sim E^{2\tau}$ sets of diameter at most $E^\rho$. Write

$$\varphi = \sum_\alpha \varphi_\alpha \text{ with } \varphi_\alpha(x) = \sum_{\xi \in \mathcal{E}_\alpha} \hat\varphi(\xi)e(x\cdot\xi)$$

and

$$\int\limits_{\Sigma} |\varphi|^4 \omega d\sigma \le E^{2\tau} \sum_\alpha \int\limits_{\Sigma} |\varphi_\alpha|^2 |\varphi|^2 \omega d\sigma. \tag{10.35}$$

We choose $\tau$ small enough for Linnik's equidistribution property to imply

$$\#\mathcal{E}_\alpha \ll E^{\frac{1}{2}-2\tau+o(1)} \text{ for each } \alpha \tag{10.36}$$

(this is where we need to assume $E \ne 0, 4, 7 \bmod 8$). Expanding in Fourier and using again (3.10), we obtain

$$\int\limits_{\Sigma} |\varphi|^2 |\varphi_\alpha|^2 \omega d\sigma$$

$$\lesssim \left| \sum_{\xi_1-\xi_2+\xi_3-\xi_4\ne 0} \hat\varphi(\xi_1)\overline{\hat\varphi(\xi_2)}\,\hat\varphi_\alpha(\xi_3)\overline{\hat\varphi_\alpha(\xi_4)}\,\frac{e^{i\psi(\xi_1-\xi_2+\xi_3-\xi_4)}}{|\xi_1-\xi_2+\xi_3-\xi_4|} \right| \tag{10.37}$$

$$+ \sum_{\xi_1,\xi_2,\xi_3,\xi_4} |\hat\varphi(\xi_1)|\,|\hat\varphi(\xi_2)|\,|\hat\varphi_\alpha(\xi_3)|\,|\hat\varphi_\alpha(\xi_4)|\,(1+|\xi_1-\xi_2+\xi_3+\xi_4|)^{-2}.$$

$$\tag{10.38}$$

From (10.34), clearly

$$(10.38) < E^{-1+o(1)} \sum_{\xi_1, \xi_2 \in \mathcal{E}; \xi_3, \xi_4 \in \mathcal{E}_\alpha} (1 + |\xi_1 - \xi_2 + \xi_3 - \xi_4|)^{-2}$$

$$\lesssim E^{-1+o(1)} \sum_{2^k < E^{\frac{1}{2}}} 4^{-k} \#\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{E}^2 \times \mathcal{E}_\alpha^2; |\xi_1 - \xi_2 + \xi_3 - \xi_4| < 2^k\}.$$

$$(10.39)$$

We need to estimate for $r < R$

$$\#\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{E}^2 \times \mathcal{E}_\alpha^2; |\xi_1 - \xi_2 + \xi_3 - \xi_4| < r\}. \tag{10.40}$$

Let $P_\delta$ be an approximate identity on $\mathbb{T}^3$. Then

$$(10.40) \lesssim \int_{\mathbb{T}^3} \left| \sum_{\xi \in \mathcal{E}} e(x.\xi) \right|^2 \left| \sum_{\xi \in \mathcal{E}_\alpha} e(x.\xi) \right|^2 P_{\frac{1}{r}}(x) dx$$

$$\lesssim r^3 \int_{\mathbb{T}^3} \left| \sum_{\xi \in \mathcal{E}} e(x.\xi) \right|^2 \left| \sum_{\xi \in \mathcal{E}_\alpha} e(x.\xi) \right|^2 dx$$

$$= r^3 \#\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{E}^2 \times \mathcal{E}_\alpha^2; \xi_1 - \xi_2 = \xi_3 - \xi_4\} \tag{10.41}$$

$$\leq r^3 |\mathcal{E}_\alpha|^2 \left[ \max_{v \in \mathbb{Z}^3} (\#\{(\xi, \eta) \in \mathcal{E}^2; \xi - \eta = v\}) \right]$$

$$\ll r^3 |\mathcal{E}_\alpha|^2 E^\varepsilon$$

$$\ll r^3 E^{1-4\tau+\varepsilon}. \tag{10.42}$$

We used here (10.42) and the bound

$$\#\{(\xi, \eta) \in \mathbb{Z}^3 \times \mathbb{Z}^3 : |\xi|^2 = E = |\eta|^2 \text{ and } \xi - \eta = v\} \ll E^\varepsilon \tag{10.43}$$

which is a consequence of (2.6).

Another bound on (10.38) is obtained by fixing $\xi_2 \in \mathcal{E}, \xi_3, \xi_4 \in \mathcal{E}_\alpha$ and observing that $\xi_1 \in \mathcal{E}$ is restricted to some ball of radius $r$. Hence, invoking Lemma 8.9,

$$(10.40) \ll |\mathcal{E}_\alpha|^2 |\mathcal{E}| E^\varepsilon \left( 1 + r \left( \frac{r}{\sqrt{E}} \right)^{\frac{1}{20}} \right)$$

$$\ll E^{\frac{3}{2}-4\tau+\varepsilon} \left( 1 + r \left( \frac{r}{\sqrt{E}} \right)^{\frac{1}{20}} \right). \tag{10.44}$$

Thus

$$(10.38) \ll E^{-4\tau+\varepsilon} \sum_{2^k < E^{\frac{1}{2}}} \min \left( 2^k, 4^{-k} E^{\frac{1}{2}} + 2^{-k} E^{\frac{1}{2}} \left( \frac{2^k}{\sqrt{E}} \right)^{\frac{1}{20}} \right) \ll E^{\frac{1}{4}-4\tau+\varepsilon}. \tag{10.45}$$

Next, we estimate (10.37). Let $0 \leq \eta \leq 1$ be a bump function on $\mathbb{R}^3$ such that $\eta(x) = 0$ if $|x| < \frac{1}{2}$ or $|x| \geq 2$. Estimate

$$(10.37) < \sum_{2^k < \sqrt{E}} \left| \sum_{\xi_1, \xi_2, \xi_3, \xi_4} \eta \left( \frac{\xi_1 - \xi_2 + \xi_3 - \xi_4)}{2^k} \right) \hat{\varphi}(\xi_1) \overline{\hat{\varphi}(\xi_2)} \hat{\varphi}_\alpha(\xi_3) \overline{\hat{\varphi}_\alpha(\xi_4)} \right.$$

$$\left. \times \frac{e^{i\psi(\xi_1 - \xi_2 + \xi_3 - \xi_4)}}{|\xi_1 - \xi_2 + \xi_3 - \xi_4|} \right|. \tag{10.46}$$

Ignoring the oscillatory factor, the $k$-term in (10.46) can be estimated by

$$E^{-1+o(1)} 2^{-k} [\#\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{E}^2 \times \mathcal{E}_\alpha^2; |\xi_1 - \xi_2 + \xi_3 - \xi_4| \lesssim 2^k\}] \tag{10.47}$$

recalling (10.34). From (10.42), (10.44)

$$(10.47) \ll E^{-4\tau+\varepsilon} \min \left( 4^k, 2^{-k} E^{\frac{1}{2}} + E^{\frac{1}{2}} \left( \frac{2^k}{\sqrt{E}} \right)^{\frac{1}{20}} \right)$$

$$\ll E^{\frac{1}{3} - 4\tau + \varepsilon} + E^{\frac{1}{2} - 4\tau + \varepsilon} \left( \frac{2^k}{\sqrt{E}} \right)^{\frac{1}{20}}. \tag{10.48}$$

This estimate is conclusive unless $2^k > E^{\frac{1}{2} - \varepsilon_1} \gg E^\rho$ ($\varepsilon_1 > 0$ an arbitrary small fixed constant). For such $k$, the oscillatory factor in (10.46) cannot be ignored.

Estimate the $k$-term in (10.46) by

$$(\#\mathcal{E}_\alpha)^2 \cdot E^{-\frac{1}{2}+\varepsilon} \max_{\xi_3, \xi_4 \in \mathcal{E}_\alpha} \left| \sum_{\xi_1, \xi_2 \in \mathcal{E}} \eta \left( \frac{\xi_1 - \xi_2 + \xi_3 - \xi_4}{2^k} \right) \hat{\varphi}(\xi_1) \overline{\hat{\varphi}(\xi_2)} \right.$$

$$\left. \times \frac{e^{i\psi(\xi_1 - \xi_2 + \xi_3 - \xi_4)}}{|\xi_1 - \xi_2 + \xi_3 - \xi_4|} \right|$$

$$\ll 2^{-k} E^{-4\tau+\varepsilon} \left\{ \max_{\substack{|v| < E^\rho \\ |a_\xi|, |b_\xi| \leq 1}} \left| \sum_{\substack{\xi_1, \xi_2 \in \mathcal{E} \\ |\xi_1 - \xi_2| > 2^{k-2}}} a_{\xi_1} b_{\xi_2} \, e^{i\psi(\xi_1 - \xi_2 + v)} \right| \right\}. \tag{10.49}$$

In establishing (10.31), we may obviously assume $\mathrm{diam}(\mathrm{supp}\,\hat{\varphi}) < c\sqrt{E}$ ($c$ as in Lemma 7.2). It remains to get a nontrivial bound on

$$\sum_{\substack{\xi_1, \xi_2 \in \mathcal{E} \\ 2^{k-2} < |\xi_1 - \xi_2| < c\sqrt{E} \\ \xi_1 - \xi_2 + v \in Z}} a_{\xi_1} b_{\xi_2} \, e^{i\psi(\xi_1 - \xi_2 + v)} \tag{10.50}$$

where $|v| < E^\rho, \rho < \frac{1}{2} - \varepsilon_1$. The same analysis used to prove Lemma 7.2 gives an estimate

$$|(10.50)| < E^{1-\gamma} \tag{10.51}$$

for some $\gamma > 0$[1]

Hence

$$(10.49) \ll E^{\frac{1}{2}-\gamma-4\tau+\varepsilon_1+\varepsilon} < E^{\frac{1}{2}-\frac{1}{2}\gamma-4\tau}. \tag{10.52}$$

Thus from (10.48), (10.52), we obtain

$$(10.37) \ll E^{\frac{1}{2}-4\tau-\frac{1}{20}\varepsilon_1+\varepsilon} + E^{\frac{1}{2}-\frac{1}{2}\gamma-4\tau} \tag{10.53}$$

and recalling (10.35)

$$\int_\Sigma |\varphi|^4 d\sigma \lesssim E^{4\tau} \times (10.53) < E^{\frac{1}{2}-\frac{1}{21}\varepsilon_1} + E^{\frac{1}{2}-\frac{1}{2}\gamma}. \tag{10.54}$$

This completes the proof of (10.31) and Theorem 10.9. $\qquad\square$

REMARK. It is easily seen that if $\Sigma \subset \mathbb{T}^3$ is a smooth surface, then

$$\max_{\|\varphi_E\|_2=1} \left( \int_\Sigma |\varphi_E|^4 d\sigma \right) \gtrsim \frac{1}{E}(\#\{\xi \in \mathbb{Z}^3; |\xi|^2 = E\})^2 \tag{10.55}$$

(consider the contribution of $\Sigma' \cap B(0, \frac{1}{10R})$ with $0 \in \Sigma'$ a shift of $\Sigma$).

Since by (2.3) there are arbitrary large eigenvalues $E$ for which

$$\#\{\xi \in \mathbb{Z}^3; |\xi|^2 = E\} \gtrsim E^{1/2}(\log \log E) \tag{10.56}$$

one cannot hope for uniform $L^4$-restriction bounds.

## 11  Higher Dimension

**11.1.**  We are not able at the time of this writing to prove either Theorem 8.1 or Theorem 9.1 in dimension $\geq 4$.

It was proven by Hu [Hu09] that if $(M, g)$ is a smooth compact Riemannian manifold of dimension $d$ and $\Sigma$ a smooth submanifold of dimension $d-1$ with positive (or negative) definite second fundamental form, then

$$\|\varphi_E\|_{L^2(\Sigma)} < C_\Sigma E^{\frac{1}{12}}\|\varphi_E\|_{L^2(M)} \tag{11.1}$$

---

[1] Letting $R = \sqrt{E}$, $v' = \frac{v}{R}, |v'| < R^{-2\tau}$, consider the function $\psi(x - y + v')$ with $x, y \in S^2$. The sets $\mathcal{W}_{\delta,\delta_1}$ considered in Lemma 4.3 for the function $\psi(p(x)-p(y))$ remain the same for the function $\psi(p(x) - p(y) + v')$, since $\delta, \delta_1 > R^{-\varepsilon}$ while $|v'| < R^{-2\tau}$, $\varepsilon < \tau$. Thus the analysis from Section 4 still applies and we obtain Lemma 5.2 for $\psi(x - y)$ replaced by $\psi(x - y + v)$.

for all eigenfunctions $\varphi_E, -\Delta\varphi_E = E\varphi_E$ of the Laplace–Beltrami operator $\Delta$ of $M$. For $d = 2$, the result is due to [BGT07].

In the case of the flat torus $M = \mathbb{T}^d$, one can show an improvement over (11.1) in arbitrary dimension

$$\|\varphi_E\|_{L^2(\Sigma)} < C_\Sigma E^{\frac{1}{12} - \varepsilon_d} \|\varphi_E\|_{L^2(\mathbb{T}^d)} \tag{11.2}$$

for some $\varepsilon_d > 0$ (with same assumption on $\Sigma$).

We will not present the proof here, as we believe the validity of our Theorem 8.1 in any dimension is the truth.

**11.2.** Theorem 8.1 in its dual formulation is the following statement about restriction of the Fourier transform.

**Theorem 11.1.** *Let $\Sigma \subset \mathbb{T}^3$ be real analytic with nowhere vanishing curvature. For $E \in \mathbb{Z}_+$, denote*

$$\mathcal{E}_E = \{\xi \in \mathbb{Z}^3; |\xi|^2 = E\}.$$

*Then the restriction operator*

$$L^2\left(\Sigma, d\sigma\right) \to \ell^2(\mathcal{E}_E) : \mu \to \hat{\mu}|_{\mathcal{E}_E}$$

*has norm bounded by $C_\Sigma$.*

Setting $R = \sqrt{E}$, our argument involves the following properties of $\mathcal{E} = \mathcal{E}_E \subset RS^2$:

(i) There is $\varepsilon_1 > 0$ such that if $r = R^{1-\varepsilon_1}$ and $C_r \subset RS^2$ is a cap of size $r$, then (for some sufficiently small $\varepsilon > 0$)

$$|\mathcal{E} \cap C_r| \lesssim \left(\frac{r}{R}\right)^2 R^{1+\varepsilon} \tag{11.3}$$

(ii) There is some constant $\eta > 0$ such that if $r < R$ and $C_r \subset RS^2$, then

$$|\mathcal{E} \cap C_r| \ll R^\varepsilon \left[\left(\frac{r}{R}\right)^\eta r + 1\right] \quad \text{for all } \varepsilon > 0 \tag{11.4}$$

(iii) Denoting $\{C_\alpha\}$ a partition of $RS^2$ in cells of size $\sim \sqrt{R}$,

$$\sum_\alpha |\mathcal{E} \cap C_\alpha|^2 \ll R^{1+\varepsilon} \quad \text{for all } \varepsilon > 0 \tag{11.5}$$

holds.

Note that we did not use the fact that $\mathcal{E} \subset \mathbb{Z}^3$.

The 'idealization' of $\mathcal{E}$ is a set $\mathcal{S} \subset RS^2$ which elements are $\sqrt{R}$-separated. For such sets $\mathcal{S}$, the restriction operator

$$L^2\left(\Sigma, d\sigma\right) \to \ell^2(\mathcal{S}) : \mu \to \hat{\mu}|_{\mathcal{S}} \tag{11.6}$$

with $\Sigma$ as in Theorem 11.1, is easily seen to be bounded. By (3.10), it suffices indeed to show that

$$\sum_{\xi,\xi'\in\mathcal{S}}\frac{|a_\xi|\,|a_{\xi'}|}{|\xi-\xi'|+1}\leq C\left(\sum_{\xi\in\mathcal{S}}|a_\xi|^2\right). \tag{11.7}$$

Our assumption on $\mathcal{S}$ implies that $\max_{\xi'}\left(\Sigma_{\xi\in S}\frac{1}{|\xi-\xi'|}\right)<C$ and (11.7) follows from Schur's test.

Surprisingly, the higher dimensional analogue, where one considers a set $\mathcal{S}\subset RS^{d-1}$ of $R^{\frac{1}{d-1}}$-separated points as idealization of $\mathcal{E}=\{\xi\in\mathbb{Z}^d;|\xi|^2=R^2\}$, may fail for $d$ large enough. This illustrates the difficulty of proving Theorem 8.1 for general dimension and the need to exploit somehow that $\mathcal{E}\subset\mathbb{Z}^d$.

In the next example $\Sigma=S^{d-1}$.

LEMMA 11.2. *Let $d\geq 8$. Then for large $R$ there is a set $S=S(R)\subset\{x\in\mathbb{R}^d|\,|x|=R\}$ with the following property*

$$|\xi-\xi'|\gtrsim R^{\frac{1}{d-1}}\quad\text{for }\xi\neq\xi'\text{ in }S \tag{11.8}$$

*and such that the operator*

$$L^2(S^{d-1},d\sigma)\to\ell^2(S):\mu\mapsto\hat{\mu}|S$$

*has norm at least $R^{\frac{1}{6}-\frac{1}{d-1}}$.*

*Proof.* Let $K=[R^{\frac{1}{d-1}}]$. In fact we will only use points in the cap

$$C=\{|x|=R\}\cap B\left(Re_d,\frac{1}{100}R^{2/3}\right)$$

(see Fig. 4). We choose

$$S=\left\{\left(Kz_1,\ldots,Kz_{d-1},\sqrt{R^2-K^2(z_1^2+\cdots+z_{d-1}^2)}\right),z_i\in\mathbb{Z},|z|<\frac{R^{2/3}}{100K}\right\}. \tag{11.9}$$

Next, we introduce the measure $\mu$ on $S_{d-1}$, $\|\frac{d\mu}{d\sigma}\|_2=1$. Let

$$\mathcal{F}=\{(y_1,\ldots,y_{d-1})\in\mathbb{Z}^{d-1};y_1^2+\cdots+y_{d-1}^2=K^2\}$$

Thus

$$|\mathcal{F}|\sim K^{d-3}.$$

Define

$$\Omega=\left\{x=(x_1,\ldots,x_d)\in S^{d-1};\operatorname{dist}\left(x',\frac{1}{K}\mathcal{F}\right)<R^{-\frac{2}{3}}\right\}$$
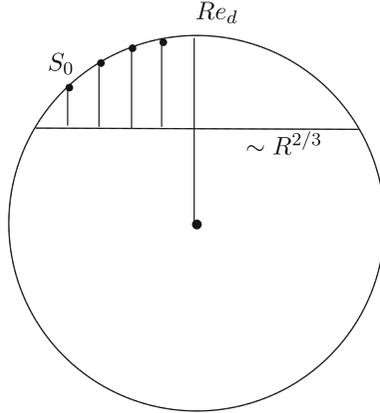
Figure 4: The cap $C$

where $x' = (x_1, \ldots, x_{d-1})$. Hence

$$1 - x_d^2 = |x'|^2 > 1 - 2R^{-2/3} \quad \text{and} \quad |x_d| < \sqrt{2}R^{-\frac{1}{3}}.$$

Also

$$|\Omega| \sim |\mathcal{F}| \cdot R^{-\frac{2}{3}(d-2)}R^{-\frac{1}{3}} \sim K^{d-3}R^{-\frac{2}{3}d+1}. \tag{11.10}$$

Define $\mu$ on $S_{d-1}$ by

$$\frac{d\mu}{d\sigma} = \frac{e(-R \cdot x_d)}{|\Omega|^{\frac{1}{2}}} 1_\Omega. \tag{11.11}$$

Evaluate

$$\sum_{\xi \in S} |\hat{\mu}(\xi)|^2 = |\Omega|^{-1} \sum_{\xi \in S} |\widehat{1_\Omega}(\xi - Re_d)|^2. \tag{11.12}$$

Note that $S - Re_d$ is contained in $\frac{1}{100}R^{2/3} \times \cdots \times \frac{1}{100}R^{2/3} \times \frac{1}{100}R^{1/3}$ and therefore, from definition of $\Omega$ and $\mathcal{S}$

$$e\big((\xi - Re_d) \cdot x\big) \approx e(\xi_1 x_1 + \cdots + \xi_{d-1}x_{d-1})$$
$$\approx e(Kz \cdot x') = 1 \tag{11.13}$$

for $\xi \in S, x \in \Omega$ and $z \in \mathbb{Z}^{d-1} \cap B\big(0, \frac{R^{2/3}}{100K}\big)$. It follows from the definition of $\Omega$ that

$$\|\hat{\mu}|_S\|_2^2 \gtrsim |\mathcal{S}| \, |\Omega| \sim \Big(\frac{R^{2/3}}{K}\Big)^{d-1} \cdot K^{d-3}R^{-\frac{2}{3}d+1} = R^{\frac{1}{3}}K^{-2} \tag{11.14}$$

hence the claim. Note that we may replace $S$ by $T(S)$, with $T$ an arbitrary orthogonal transformation of $\mathbb{R}^d$, with the same conclusion for the restriction operator. $\qquad \square$

## 12  Restriction Upper Bounds for Generic Eigenvalues

In this section we prove Theorem 1.10. The proof of Theorem 1.10 is based on the following arithmetic statement (Lemma 2.9 in [BR11b]).

LEMMA 12.1. *Fix $\varepsilon > 0$ and taking $N \in \mathbb{Z}_+$ large, $E \in \{1, \dots, N\}$ and $\lambda = \sqrt{E}$, one has that*

$$\min_{\substack{x \neq y \in \mathbb{Z}^2 \\ |x| = \lambda = |y|}} |x - y| > \lambda^{1-\varepsilon} \tag{12.1}$$

*except for a set of $E$-values of size at most $N^{1-\frac{\varepsilon}{3}}$.*

We recall the argument.

*Proof of Lemma 12.1.* Let $M = \sqrt{N}$ and estimate the size of the set

$$S = \{x \in \mathbb{Z}^2; |x| \leq M \text{ and } 2x.z = |z|^2 \text{ for some } z \in \mathbb{Z}^2, 0 < |z| < M^{1-\varepsilon}\}. \tag{12.2}$$

Writing $z = d.z', d \in \mathbb{Z}_+$ and $z' = (z_1', z_2') \in \mathbb{Z}^2$ primitive, the equation

$$2x.z' = d|z'|^2 \tag{12.3}$$

has at most $c\frac{M}{|z'|}$ solutions in $x, |x| \leq M$, for given $z'$ primitive.

Hence

$$|S| \leq C \sum_{1 \leq d < M} \sum_{\substack{z' \in \mathbb{Z}^2 \\ 0 < |z'| < \frac{M^{1-\varepsilon}}{d}}} \frac{M}{|z'|} < C \sum_{d < M} \frac{M^{2-\varepsilon}}{d} < CM^{2-\varepsilon} \log N.$$

Since $|S|$ is obviously an upper bound for the number of exceptional $E \in \{1, \dots, N\}$, Lemma 12.1 follows.

Theorem 1.10 is therefore a consequence of

LEMMA 12.2. *Let $\varepsilon > 0$ be small enough and $E = \lambda^2 \in \mathbb{Z}_+$ satisfy (12.1).*
*Let $\Sigma$ be a $C^2$-smooth curve in $\mathbb{T}^2$. Then any eigenfunction $\varphi_\lambda$ of $\mathbb{T}^2$ satisfies*

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \leq C_\Sigma \|\varphi_\lambda\|_2. \tag{12.4}$$

*Proof.* Let $\gamma : I \to \Sigma$, $I \subset [0,1]$, be an arclength parametrization. Fix $\frac{1}{2} < \rho < 1$ and partition $I = \bigcup I_s, I_s = [t_s, t_{s+1}]$ in intervals of size $\lambda^{-\rho}$. Since

$$\gamma(t) = \gamma(t_s) + (t - t_s)\dot{\gamma}(t_s) + O(\lambda^{-2\rho})$$

for $t \in I_s$, it follows that

$$\left| \int_I e^{i\xi \cdot \gamma(t)} dt \right| \leq \sum_s \left| \int_{t_s}^{t_{s+1}} e^{i\xi \cdot \gamma(t)} dt \right|$$

$$= \sum_s \left| \int_0^{\lambda^{-\rho}} e^{i\xi \cdot \dot{\gamma}(t_s)u} du \right| + O(|\xi|\lambda^{-2\rho}). \tag{12.5}$$

Denote $\mathcal{E} = \{\xi \in \mathbb{Z}^2; |\xi| = \lambda\}$ and $\varphi = \sum_{\xi \in \mathcal{E}} a_\xi e(x \cdot \xi)$, $\|\varphi\|_2 \leq 1$. Estimate using (12.5)

$$
\int_\Sigma |\varphi|^2 d\sigma \leq \sum_{\xi,\xi' \in \mathcal{E}} |a_\xi| \, |a_{\xi'}| \left| \int_I e\big(\gamma(t) \cdot (\xi - \xi')\big) dt \right|
$$

$$
\leq \sum_s \sum_{\xi,\xi' \in \mathcal{E}} |a_\xi| \, |a_{\xi'}| \min\left\{ \lambda^{-\rho}, \frac{1}{|(\xi - \xi') \cdot \dot{\gamma}(t_s)|} \right\} + |\mathcal{E}|^2 \lambda^{1-2\rho}. \quad (12.6)
$$

Fix $1 \leq s \leq \lambda^\rho$. If we fix $\xi \in \mathcal{E}$ and let $\xi' \in \mathcal{E} \backslash \{\xi\}$ vary, it follows from (12.1) that

$$
|P_{\dot{\gamma}(t_s)}(\xi - \xi')| \gtrsim \lambda^{-\varepsilon} |\xi - \xi'| \gtrsim \lambda^{1-2\varepsilon} \quad (12.7)
$$

except for at most 1 element. Thus (12.7) holds for $(\xi, \xi') \notin \mathcal{E}_s$ where $\mathcal{E}_s \subset \mathcal{E}$ has the property that for each $\xi$ (resp. $\xi'$) there is at most one $\xi'$ (resp. $\xi$) with $(\xi, \xi') \in \mathcal{E}_s$. From (12.7)

$$
(12.6) \lesssim \sum_s \sum_{\xi,\xi' \in \mathcal{E}} |a_\xi| \, |a_{\xi'}| \lambda^{-1+2\varepsilon} + \sum_s \lambda^{-\rho} \sum_{(\xi,\xi') \in \mathcal{E}_s} |a_\xi| \, |a_{\xi'}| + |\mathcal{E}|^2 \lambda^{1-2\rho}
$$

$$
\lesssim \lambda^{-1+\rho+2\varepsilon} |\mathcal{E}|^2 + \max_s \sum_{(\xi,\xi') \in \mathcal{E}_s} |a_\xi| \, |a_{\xi'}| + |\mathcal{E}|^2 \lambda^{1-2\rho}
$$

$$
\lesssim 1 + |\mathcal{E}|^2 (\lambda^{-1+\rho+2\varepsilon} + \lambda^{1-2\rho}) \lesssim 1
$$

for $\varepsilon > 0$ small enough, since $\frac{1}{2} < \rho < 1$.

This proves Lemma 12.2.                                                          □

## 13   The Number of Nodal Domains for a Random Eigenfunction

In this section we prove the analogue of the Nazarov–Sodin theorem [NS09] on the number of nodal domains for $\mathbb{T}^d$, $d \geq 3$. We restrict ourselves to $d = 3$ as some extra arithmetical assumptions are required in this case.

**Theorem 13.1.** *Let $d = 3$. Assume $E = \lambda^2 \in \mathbb{Z}$ sufficiently large and $E \neq 0, 4, 7 (mod\, 8)$. The number of components of the nodal set $N$ of a 'typical' eigenfunction $\varphi_\lambda$ is of the order $\lambda^3$.*

*Proof.* In [NS09], the corresponding result is proven for the sphere, based on a 'barrier' argument. It turns out that the same method can be easily adapted to the

torus $\mathbb{T}^d$, at least when $d \geq 3$, to produce the required lower bound (the upper bound follows from Courant's nodal domain theorem).

First, denoting $X_\lambda = \text{span}\{\varphi; -\Delta\varphi = \lambda^2\varphi\}$ and $P(X_\lambda)$ the corresponding projective space, a generic element of $P(X_\lambda)$ is represented by a Gaussian random variable

$$\varphi^\omega(x) = \frac{1}{|\mathcal{E}|^{\frac{1}{2}}} \sum_{\xi \in \mathcal{E}} \left( g_\xi(\omega) \cos 2\pi x \cdot \xi + h_\xi(\omega) \sin 2\pi x \cdot \xi \right) \tag{13.1}$$

with $\mathcal{E} \cap (-\mathcal{E}) = \phi$, $\mathcal{E} \cup (-\mathcal{E}) = \{\xi \in \mathbb{Z}^d : |\xi| = \lambda\}$ and $\{g_\xi\}, \{h_\xi\}$ independent, real, normalized Gaussian random variables.

Denoting

$$\{f_j\} = \{\sqrt{2}\cos 2\pi x \cdot \xi, \quad \sqrt{2}\sin 2\pi x \cdot \xi : \xi \in \mathcal{E}\} \tag{13.2}$$

rewrite $\varphi^\omega$ as

$$\varphi^\omega = \frac{1}{\sqrt{2|\mathcal{E}|}} \sum_{1 \leq j \leq 2|\mathcal{E}|} g_j(\omega)f_j \tag{13.3}$$

where $\{g_j\}$ are as above.

Denote $N = 2|\mathcal{E}|$ and let $T$ be an $N \times N$ orthogonal matrix. Defining

$$F_i(x) = \sum_{j=1}^{N} T_{ij}f_j(x) \tag{13.4}$$

the Gaussian random variable $\varphi^\omega$ has the same distribution as

$$\psi^\omega = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} g_j(\omega)F_j \tag{13.5}$$

(by invariance of the Gaussian ensemble under the orthogonal group).

Choose $T$ with

$$T_{1j} = \begin{cases} \frac{1}{\sqrt{|\mathcal{E}|}} & \text{if } f_j \text{ is even} \\ 0 & \text{if } f_j \text{ is odd.} \end{cases} \tag{13.6}$$

Hence

$$F_1(x) = \frac{\sqrt{2}}{\sqrt{|\mathcal{E}|}} \sum_{\xi \in \mathcal{E}} \cos 2\pi x \cdot \xi \tag{13.7}$$

that we use as our 'barrier' function.

Rewrite

$$\psi^\omega = \frac{1}{\sqrt{N}} g_1(\omega)F_1 + G^\omega \tag{13.8}$$

with $G^\omega$ independent of $g_1$.

Taking in (13.7) $\|x\| \lesssim \lambda^{-1}$, it follows from the equidistribution of lattice points on the sphere (this is why we impose the condition $E \neq 0, 4, 7 \mod 8$, see Sect. 2.1) that

$$F_1(x) = \sqrt{N} \left\{ \int_{S^2} (\cos 2\pi \lambda x \cdot \zeta) \sigma(d\zeta) + O(\lambda^{-\varepsilon}) \right\}$$

$$= \sqrt{N} \big( \hat{\sigma}(\lambda |x|) + O(\lambda^{-\varepsilon}) \big). \tag{13.9}$$

Therefore, there is some $r \sim \frac{1}{\lambda}$ such that (for some constant $c > 0$)

$$F_1(x) < -c\sqrt{N} \text{ for } |x| = r. \tag{13.10}$$

Also, clearly

$$F_1(0) = \sqrt{N}. \tag{13.11}$$

Assume we show that for some constant $C_1$,

$$\max_{|x| \leq r} |G^\omega(x)| < C_1 \tag{13.12}$$

holds with probability at least $\frac{1}{2}$ in $\omega$.

Since $g_1(\omega)$ is independent of $G^\omega$, it follows from (13.10), (13.11)

$$\psi^\omega(0) \geq g_1(\omega) - |G^\omega(0)| > C_2 - C_1 > 1 \tag{13.13}$$

and for $|x| = r$

$$\psi^\omega(x) < -cg_1(\omega) + \max_{|x|=r} |G^\omega(x)| < -cC_2 + C_1 < -1 \tag{13.14}$$

with probability at least $\frac{1}{2} e^{-C_2^2} > c_3 > 0$ in $\omega$. For such $\omega$, since $\psi^\omega$ satisfies (13.13), (13.14), the ball $B(x,r) \subset \mathbb{T}^3$ will necessarily contain a nodal component.

Partitioning $\mathbb{T}^3$ in boxes $Q_\alpha$ of size $\sim \frac{1}{\lambda}$ and observing that $\varphi^\omega$ and any translate $\varphi^\omega(\cdot + a), a \in \mathbb{T}^3$, are random variables with the same distribution, the preceding implies that, with large probability in $\omega$, the nodal set $N_\omega$ of $\varphi^\omega$ satisfies

$$\#\{\alpha; Q_\alpha \text{ contains a component of } N_\omega\} \sim \lambda^3$$

and hence $\varphi^\omega$ has at least $\sim \lambda^3$ nodal components.

It remains to justify (13.12).

Take a radial bumpfunction $\eta$ on $\mathbb{R}^3$ such that

$$\eta(x) \sim e^{-|x|} \quad \text{and} \quad \hat{\eta}(x) = 1 \text{ for } |x| = 1 \tag{13.15}$$

and set $\eta_\lambda(x) = \lambda^3 \eta(\lambda x)$. Thus $\hat{\eta}_\lambda(x) = 1$ for $|x| = \lambda$ and therefore

$$G^\omega = G^\omega * \eta_\lambda. \tag{13.16}$$

Since $\int \eta_\lambda = \int \eta < C$, clearly

$$\int_{\mathbb{R}^3} |G^\omega(x)| \eta_\lambda(x) dx < C \tag{13.17}$$

with probability at least $\frac{1}{2}$ in $\omega$.

Let $|y| \leq r$. By (13.16), $|G^\omega(y)| \leq \int |G^\omega(x)| \eta_\lambda(x - y) dx$, and since $\eta_\lambda(x - y) \sim \eta_\lambda(x)$ for $|y| \lesssim \frac{1}{\lambda}$ by the choice of $\eta$ in (13.15), (13.12) follows from (13.17).

This completes the proof of Theorem 13.1.                    $\square$

## Acknowledgements

## Appendix A.  Lattice Points in Caps ($d \geq 4$)

Let $N = R^2 \in \mathbb{Z}$. We show the following

$$|\mathcal{E}_R \cap C_r| \lesssim \begin{cases} \frac{r^{d-1}}{R} + r^{d-3} & \text{if} \quad d \geq 7 \\[2mm] \frac{r^5}{R} + (\log \omega(N))^2 r^3 & \text{if} \quad d = 6 \\[2mm] \frac{r^4}{R} + r^{2+\varepsilon} R^\varepsilon & \text{if} \quad d = 5 \\[2mm] \frac{r^3}{R}(\log \omega(N))^2 + r^{\frac{3}{2}+\varepsilon} R^\varepsilon & \text{if} \quad d = 4. \end{cases} \tag{A.1}$$

Let $N = R^2$ and $b = (b_1, \ldots, b_d) \in \mathcal{E} \cap C_r$. Then

$$|\mathcal{E}_R \cap C_r| \leq |\{x \in \mathbb{Z}^d | x_1^2 + \cdots + x_d^2 = N \quad \text{and} \quad |x_j - b_j| \leq r\}|$$
$$\leq \left| \left\{ y \in \mathbb{Z}^d \cap B_r \Big| \sum_{j+1}^d y_j^2 + 2b_j y_j = 0 \right\} \right|. \tag{A.2}$$

Let $\gamma$ be a smooth bump function. Express (A.2) by the circle method as

$$\int_{\mathbb{T}} \prod_{j=1}^d \left[ \sum \gamma\left(\frac{y}{r}\right) e\big((y^2 + 2b_j y)t\big) \right] dt. \tag{A.3}$$

Denote

$$G(t, \varphi) = \sum_y \gamma\left(\frac{y}{r}\right) e\left(y^2 t + y\varphi\right). \tag{A.4}$$

Let

$$t = \frac{a}{q} + \beta, q < r, (a, q) = 1 \quad \text{and} \quad |\beta| < \frac{1}{qr}. \tag{A.5}$$

By Poisson summation

$$G(t, \varphi) \sim \sum_{m \in \mathbb{Z}} S(a, m; q) J(\varphi, \beta, m; q) \tag{A.6}$$

where

$$S(a, m; q) = \frac{1}{q} \sum_{k=0}^{q-1} e_q(k^2 a - km) \tag{A.7}$$

and

$$J(\varphi, \beta, m; q) = \int_{\mathbb{R}} \gamma\left(\frac{y}{r}\right) e\left(\left(\varphi + \frac{m}{q}\right)y + y^2\beta\right) dy. \tag{A.8}$$

Note that certainly

$$|J(\varphi, \beta, m; q)| \lesssim \min\left(r, \frac{1}{\sqrt{|\beta|}}\right) \tag{A.9}$$

and also (for appropriate choice of $\gamma$)

$$|J| \lesssim r\, e^{-(r|\varphi + \frac{m}{q}|)^{1/2}} \quad \text{if} \ \left|\varphi + \frac{m}{q}\right| > 2r|\beta|. \tag{A.10}$$

In particular, it follows from (A.10) that (A.6) only involves a few significant terms.

Substitution of (A.6) in (A.3) gives

$$\sum_{m_1,\ldots,m_d} \left\{\prod_{j=1}^{d} S(a, m_j - 2ab_j, q)\right\} \left\{\prod_{j=1}^{d} J(2b_j\beta, \beta, m_j; q)\right\} \tag{A.11}$$

where it remains to perform the sum over $(a; q) = 1$, integrate in $|\beta| < \frac{1}{rq}$ and sum over $q < r$.

Since

$$S(a, m; q) = S(1, 0, q)\left(\frac{a}{q}\right) e_q(m^2 a') \quad a'a \equiv 1 (\text{mod } q) \tag{A.12}$$

the first factor in (A.11) equals

$$S(1, 0, q)^d \left(\frac{a}{q}\right)^d e_q\left(a'\left(\sum_j (m_j - 2ab_j)^2\right)\right) \sim S(1, 0, q)^d \left(\frac{a}{q}\right)^d e_q(4aN + a'|m|^2). \tag{A.13}$$

Summing (A.13) over $a$, $(a, q) = 1$ (the sum factors over the prime factorization of $q$) and applying Weil's bound on the Kloosterman sum ($d$ even) or Salié sum ($d$ odd), gives the bound

$$q^{-\frac{d}{2}} \cdot \begin{cases} \sqrt{q}\tau(q) & d \ \text{odd} \\ \sqrt{q}\tau(q)(q, N)^{\frac{1}{2}} & d \ \text{even}. \end{cases} \tag{A.14}$$

Hence

$$(\text{A.3}) \leq \sum_{q \leq r} \frac{(q, N)^{\frac{1}{2}}\tau(q)}{q^{\frac{d-1}{2}}} \sum_{m_1,\ldots,m_d} \int \prod_{j=1}^{d} |J(2b_j\beta, \beta, m_j; q)| d\beta. \tag{A.15}$$

Since $|b| = R$, we may assume $|b_1| \sim R$.

From (A.9), (A.10))

$$\sum_m |J(\varphi, \beta, m; q)| \lesssim \frac{1}{\sqrt{\beta}} + re^{-(\frac{r}{q})^{\frac{1}{2}}} \lesssim \frac{1}{\sqrt{\beta}} \tag{A.16}$$

since $|\beta| < \frac{1}{rq}$. Hence

$$(A.15) \leq \sum_{q \leq r} \frac{(q,N)^{\frac{1}{2}} \tau(q)}{q^{\frac{d-1}{2}}} \sum_{m_1} \int \frac{|J(2b_1\beta, \beta, m_1, q)|}{(\sqrt{\beta} + \frac{1}{r})^{d-1}} d\beta. \tag{A.17}$$

From (A.10)

$$|J(2b_1\beta, \beta, m_1, q)| < r\, e^{-(r|2b_1\beta + \frac{m_1}{q}|)^{\frac{1}{2}}} \text{ if } \left|2b_1\beta + \frac{m_1}{q}\right| > 2r|\beta| \tag{A.18}$$

and hence

$$\sum_{m_1} |J(2b_1\beta, \beta, m_1, q)| < r\, e^{-(\frac{r}{q}\|2b_1q\beta\|)^{\frac{1}{2}}} \text{ if } \|2b_1q\beta\| > 2rq.|\beta|. \tag{A.19}$$

We use this property to get a better estimate.
   Write

$$\beta = \frac{\ell}{2b_1q} + \beta', |\beta'| < \frac{1}{4|b_1|q} \quad \text{and} \quad \ell \in \mathbb{Z}, |\ell| \lesssim \frac{|b_1|}{r} \sim \frac{R}{r}. \tag{A.20}$$

Thus (A.19) implies

$$\sum_m |J(2b_1\beta, \beta, m; q)| \lesssim r\, e^{-(rR|\beta'|)^{\frac{1}{2}}} \text{ if } |\beta'| > 10\frac{r\ell}{R^2q}. \tag{A.21}$$

Contribution of $|\beta| < \frac{1}{r^2}$:
For such $\beta$, from (A.20), $|\ell| \lesssim \frac{Rq}{r^2}$ and (A.21) will hold if $|\beta'| \gg \frac{1}{Rr}$.
Since for $|\beta'| \lesssim \frac{1}{Rr}$, (A.21) is certainly true, it is always valid.
The $m_1$-sum in (A.17) is therefore bounded by

$$r^{d-1}\left(1 + \frac{Rq}{r^2}\right) r \int e^{(-rR|\beta'|)^{\frac{1}{2}}} d\beta'$$

$$< r^{d-1}\left(\frac{1}{R} + \frac{q}{r^2}\right). \tag{A.22}$$

This gives the contribution

$$\frac{r^{d-1}}{R}\left(\sum_{q \leq r} \frac{\tau(q)(q,N)^{\frac{1}{2}}}{q^{\frac{d-1}{2}}}\right) + r^{d-3}\left(\sum_{q \leq r} \frac{\tau(q)(q,N)^{\frac{1}{2}}}{q^{\frac{d-3}{2}}}\right). \tag{A.23}$$

Contribution of $|\beta| > \frac{1}{r^2}$:
   Let $|\beta| \sim \frac{B}{r^2}$ with $B < \frac{r}{q}$. Then $|\ell| \sim \frac{RqB}{r^2}$ and (A.21) will hold if $|\beta'| \gtrsim \frac{B}{rR}$.
Using also (A.16) the contribution in (A.17) is at most

$$\sum_{q \leq r} \frac{(q,N)^{\frac{1}{2}} \tau(q)}{q^{\frac{d-1}{2}}} \left(\frac{r^2}{B}\right)^{\frac{d-1}{2}} \left(1 + \frac{RqB}{r^2}\right)\left(\frac{B}{rR}\frac{r}{\sqrt{B}} + r \int e^{-r(R|\beta'|)^{\frac{1}{2}}} d\beta'\right)$$

$$\leq \frac{r^{d-1}}{R} B^{-\frac{d-2}{2}} \sum_{q \leq r} \frac{(q,N)^{\frac{1}{2}} \tau(q)}{q^{\frac{d-1}{2}}} + r^{d-3} \sum_{q \leq r} \frac{\tau(q)(q,N)^{\frac{1}{2}}}{q^{\frac{d-3}{2}}} B^{-\frac{d-4}{2}}. \tag{A.24}$$

Summing (A.24) over dyadic values of $B < \frac{r}{q}$ gives (A.23), except if $d = 4$, where in the second sum there is an additional $\log \frac{r}{q}$ factor.

It remains to estimate the $q$-sums in (A.23)

$$\sum_{q \le r} \frac{(q,N)^{\frac{1}{2}} \tau(q)}{q^{\frac{d-1}{2}}} \le \left( \sum_{\substack{c|N \\ c \le r}} \frac{\tau(c)}{c^{\frac{d}{2}-1}} \right) \left( \sum_{q_1 < r} \frac{1}{q_1^{\frac{d-1}{2}-\varepsilon}} \right) < \begin{cases} C & \text{for } d \ge 5 \\ C \left( \log \omega(N) \right)^2 & \text{for } d = 4 \end{cases} \quad (A.25)$$

and

$$\sum_{q \le r} \frac{(q,N)^{\frac{1}{2}} \tau(q)}{q^{\frac{d-3}{2}}} \le \left( \sum_{\substack{c|N \\ c \le r}} \frac{\tau(c)}{c^{\frac{d}{2}-2}} \right) \left( \sum_{q_1 < r} \frac{\tau(q_1)}{q_1^{\frac{d-3}{2}}} \right) \ll \begin{cases} C & \text{for } d \ge 7 \\ C \left( \log \omega(N) \right)^2 & \text{for } d = 6 \\ R^\varepsilon & \text{for } d = 5 \end{cases} \quad (A.26)$$

while for $d = 4$, we have

$$\sum_{q \le r} \frac{(q,N)^{\frac{1}{2}} \tau(q)}{q^{\frac{1}{2}}} \log \frac{r}{q} \ll r^{\frac{1}{2}+\varepsilon} R^\varepsilon. \quad (A.27)$$

This gives (A.1).

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